FRAGMENTED DEFORMATIONS OF PRIMITIVE MULTIPLE CURVES

JEAN–MARC DRÉZET

Resume. A *primitive multiple curve* is a Cohen-Macaulay irreducible projective curve Y that can be locally embedded in a smooth surface, and such that Y_{red} is smooth.

This paper studies the deformations of Y to curves with smooth irreducible components, when the number of components is maximal (it is then the multiplicity n of Y).

We are particularly interested in deformations to n disjoint smooth irreducible components, which are called *fragmented deformations*. We describe them completely. We give also a characterization of primitive multiple curves having a fragmented deformation.

Summary

Mathematics Subject Classification: 14M05, 14B20

1. Introduction

A *primitive multiple curve* is an algebraic variety Y over $\mathbb{C}$ which is Cohen-Macaulay, such that the induced reduced variety $C = Y_{red}$ is a smooth projective irreducible curve, and that every closed point of Y has a neighborhood that can be embedded in a smooth surface. These curves have been defined and studied by C. Bănică and O. Forster in [1]. The simplest examples are infinitesimal neighborhoods of projective smooth curves embedded in a smooth surface (but most primitive multiple curves cannot be globally embedded in smooth surfaces, cf. [2], theorem 7.1).

Let Y be a primitive multiple curve with associated reduced curve C, and suppose that $Y \neq C$. Let $\mathcal{I}_C$ be the ideal sheaf of C in Y. The *multiplicity* of Y is the smallest integer n such that $\mathcal{I}_C^n = 0$. We have then a filtration

$$C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$$

where C_i is the subscheme corresponding to the ideal sheaf $\mathcal{I}_C^i$ and is a primitive multiple curve of multiplicity i. The sheaf $L = \mathcal{I}_C/\mathcal{I}_C^2$ is a line bundle on C, called the *line bundle on C associated to Y*.

The deformations of double (i.e. of multiplicity 2) primitive multiple curves (also called *ribbons*) to smooth projective curves have been studied in [11]. In this paper we are interested in deformations of primitive multiple curves $Y = C_n$ of any multiplicity $n \geq 2$ to reduced curves having exactly n components which are smooth (n is the maximal number of components of deformations of Y). In this case the number of intersection points of two components is exactly $-\deg(L)$. We give some results in the general case (no assumption on $\deg(L)$) and treat more precisely the case $\deg(L) = 0$, i.e. deformations of Y to curves having exactly n disjoint irreducible components.

1.1. *Motivation* – Let $\pi : \mathcal{C} \to S$ be a flat projective morphism of algebraic varieties, P a closed point of S such that $\pi^{-1}(P) \simeq Y$, $\mathcal{O}_\mathcal{C}(1)$ a very ample line bundle on $\mathcal{C}$ and P a polynomial in one variable with rational coefficients. Let

$$\tau : \mathcal{M}_{\mathcal{O}_\mathcal{C}(1)}(P) \longrightarrow S$$

be the corresponding relative moduli space of semi-stable sheaves (parametrizing the semi-stables sheaves on the fibers of π with Hilbert polynomial P with respect to the restriction of $\mathcal{O}_\mathcal{C}(1)$, cf. [15]).

We suppose first that there exists a closed point $s \in S$ such that $\mathcal{C}_s$ is a smooth projective irreducible curve. Then in general τ is not flat (some other examples on non flat relative moduli spaces are given in [13]). The reason is that the generic structure of torsion free sheaves on Y is more complicated than on smooth curves, and some of these sheaves cannot be deformed to sheaves on the smooth fibers of π.

A coherent sheaf on a smooth algebraic variety X is locally free on some nonempty open subset of X. This is not true on Y. But a coherent sheaf E on Y is *quasi locally free* on some nonempty open subset of Y, i.e. on this open subset, E is locally isomorphic to a sheaf of the form $\bigoplus_{1\leq i\leq n} m_i\mathcal{O}_{C_i}$, the sequence of non negative integers $(m_1, \ldots, m_n)$ being uniquely determined (cf. [3], [6]). It is not hard to see that if E can be extended to a coherent sheaf on $\mathcal{C}$, flat on S, then $R(E) = \sum_{i=1} i.m_i$ must be a multiple of n. For example, it is impossible to deform the stable sheaf $\mathcal{O}_{C_i}$ on Y in sheaves on the smooth fibers, if $1 \leq i < n$.

Now suppose that all the fibers $\pi^{-1}(s)$, $s \neq P$, are reduced with exactly n smooth components. I conjecture that (with suitable hypotheses) a torsion free coherent sheaf on Y can be extended to a coherent sheaf on $\mathcal{C}$, flat on S, using the fact that we allow coherent sheaves of the reducible fibers $\mathcal{C}_s$ that have not the same rank on all the components. This would be a step in the study of the flatness of τ. For example (for suitable π), there exists a coherent sheaf $\mathcal{E}$ on $\mathcal{C}$, flat on S, such that $\mathcal{E}_P = \mathcal{O}_C$, and that for $s \neq P$, $\mathcal{E}_s$ is the structural sheaf of an irreducible component of $\mathcal{C}_s$.

Moduli spaces of sheaves on reducible curves have been studied in [16], [17], [18].

1.2. *Maximal reducible deformations* – Let (S, P) be the germ of a smooth curve. Let Y be a primitive multiple curve of multiplicity $n \geq 2$ and $k > 0$ an integer. Let $\pi : \mathcal{C} \to S$ be a flat morphism, where $\mathcal{C}$ is a reduced algebraic variety, such that

– For every closed point $s \in S$ such that $s \neq P$, the fiber $\mathcal{C}_s$ has k irreducible components, which are smooth and transverse, and any three of these components have no common point.
– The fiber $\mathcal{C}_P$ is isomorphic to Y.

We show that by making a change of variable, i.e. by considering a suitable germ (S', P') and a non constant morphism $\tau : S' \to S$, and replacing π with $\tau^*\mathcal{C} \to S'$, we can suppose that $\mathcal{C}$ has exactly k irreducible components, inducing on every fiber $\mathcal{C}_s$, $s \neq P$ the k irreducible components of $\mathcal{C}_s$. In this case π is called a *reducible deformation of* Y *of length* k.

We show that $k \leq n$. We say that π (or $\mathcal{C}$) is a *maximal reducible deformation of* Y if $k = n$.

Suppose that π is a maximal reducible deformation of Y. We show that if $\mathcal{C}'$ is the union of $i > 0$ irreducible components of $\mathcal{C}$, and $\pi' : \mathcal{C}' \to S$ is the restriction of π, then $\pi'^{-1}(P) \simeq C_i$, and π' is a maximal reducible deformation of C_i. Let $s \in S\backslash\{P\}$. We prove that the irreducible components of $\mathcal{C}_s$ have the same genus as C. Moreover, if D_1, D_2 are distinct irreducible components of $\mathcal{C}_s$, then $D_1 \cap D_2$ consists of $-\deg(L)$ points.

1.3. *Fragmented deformations (definition)* – Let Y be a primitive multiple curve of multiplicity $n \geq 2$ and $\pi : \mathcal{C} \to S$ a maximal reducible deformation of Y. We call it a *fragmented deformation of* Y if $\deg(L) = 0$, i.e. if for every $s \in S\backslash\{P\}$, $\mathcal{C}_s$ is the disjoint union of n smooth curves. In this case $\mathcal{C}$ has n irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_n$ which are smooth surfaces.

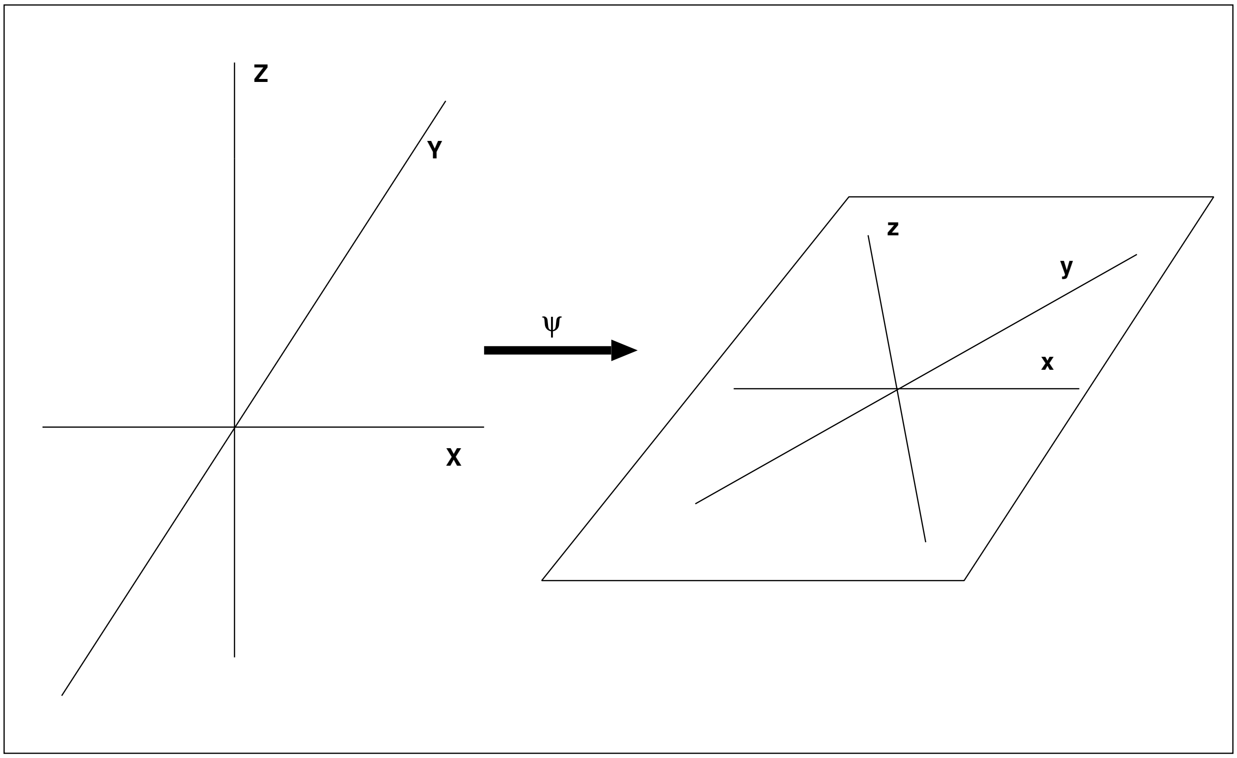
The variety $\mathcal{C}$ appears as a particular case of a *gluing* of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ along C (cf. 4.1.5). We prove (proposition 4.1.6) that such a gluing $\mathcal{D}$ is a fragmented deformation of a primitive multiple curve if and only if every closed point in C has a neighborhood in $\mathcal{D}$ that can be embedded in a smooth variety of dimension 3. The simplest gluing is the trivial or *initial gluing* $\boldsymbol{\mathcal{A}}$. An open subset U of $\boldsymbol{\mathcal{A}}$ (and $\mathcal{C}$) is given by open subsets $U_1, \ldots, U_n$ of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ respectively, having the same intersection with C, and

$$\mathcal{O}_{\boldsymbol{\mathcal{A}}}(U) \;=\; \{(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}_1}(U \cap \mathcal{C}_1) \times \cdots \times \mathcal{O}_{\mathcal{C}_n}(U \cap \mathcal{C}_n); \quad \alpha_{1|C} = \cdots = \alpha_{n|C}\},$$

and $\mathcal{O}_{\mathcal{C}}(U)$ appears as a subalgebra of $\mathcal{O}_{\boldsymbol{\mathcal{A}}}(U)$, hence we have a canonical morphism $\boldsymbol{\mathcal{A}} \to \mathcal{C}$.

We can view elements of $\mathcal{O}_{\mathcal{C}}(U)$ as n-tuples $(\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in \mathcal{O}_{\mathcal{C}_i}(U \cap \mathcal{C}_i)$. In particular we can write $\pi = (\pi_1, \ldots, \pi_n)$.

1.4. *A simple analogy* – Consider n copies of $\mathbb{C}$ glued at 0. Two extreme examples appear: the trivial gluing $\boldsymbol{\mathcal{A}}_0$ (the set of coordinate lines in $\mathbb{C}^n$), and a set $\mathcal{C}_0$ of n lines in $\mathbb{C}^2$. We can easily construct a bijective morphism $\boldsymbol{\Psi} : \boldsymbol{\mathcal{A}}_0 \to \mathcal{C}_0$ sending each coordinate line to a line in the plane



But the two schemes are of course not isomorphic: the maximal ideal of 0 in $\boldsymbol{\mathcal{A}}_0$ needs n generators, but 2 are enough for the maximal ideal of 0 in $\mathcal{C}_0$.

Let $\pi_{\mathcal{C}_0} : \mathcal{C}_0 \to \mathbb{C}$ be a morphism sending each component linearly onto $\mathbb{C}$, and $\pi_{\boldsymbol{\mathcal{A}}_0} = \pi_{\mathcal{C}_0} \circ \boldsymbol{\Psi} : \boldsymbol{\mathcal{A}}_0 \to \mathbb{C}$. The difference of $\boldsymbol{\mathcal{A}_0}$ and $\mathcal{C}_0$ can be also seen by using the fibers of 0: we have

$$\pi_{\mathcal{C}_0}^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t]/(t^n)) \quad \text{and} \quad \pi_{\boldsymbol{\mathcal{A}}_0}^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t_1,\ldots,t_{n-1}]/(t_1,\ldots,t_{n-1})^2)\ .$$

Let $\mathcal{D}$ be a general gluing of n copies of $\mathbb{C}$ at 0, such that there exists a morphism $\pi : \mathcal{D} \to \mathbb{C}$ inducing the identity on each copy of $\mathbb{C}$. It is easy to see that we have $\pi^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t]/(t^n))$ if and only if some neighborhood of 0 in $\mathcal{D}$ can be embedded in a smooth surface.

1.5. *Fragmented deformations (main properties)* – Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of $Y = C_n$. Let $I \subset \{1,\ldots,n\}$ be a proper subset, I^c its complement, and $\mathcal{C}_I \subset \mathcal{C}$ the subscheme union of the $\mathcal{C}_i, i \in I$. We prove (theorem 4.3.7) that the ideal sheaf $\mathcal{I}_{\mathcal{C}_I}$ of $\mathcal{C}_I$ is isomorphic to $\mathcal{O}_{\mathcal{C}_{I^c}}$.

In particular, the ideal sheaf $\mathcal{I}_{\mathcal{C}_i}$ of $\mathcal{C}_i$ is generated by a single regular function on $\mathcal{C}$. We show that we can find such a generator such that for $1 \leq j \leq n$, $j \neq i$, its j-th coordinate can be written as $\alpha_j \pi_j^{p_{ij}}$, with $p_{ij} > 0$ and $\alpha_j \in H^0(\mathcal{O}_S)$ such that $\alpha_j(P) \neq 0$. If $1 \leq j \leq n$ and $j \neq i$, we can then obtain a generator that can be written as

$$\mathbf{u}_{ij} \ = \ (u_1,\ldots,u_m),$$

with

$$u_i = 0, \quad u_m \ = \ \alpha_{ij}^{(m)} \pi_m^{p_{im}} \ \text{ for } m \neq i, \quad \alpha_{ij}^{(j)} = 1.$$

The constants $\mathbf{a}_{ij}^{(m)} = \alpha_{ij|C}^{(m)} \in \mathbb{C}$ have interesting properties (propositions 4.5.2, 4.4.6). Let $p_{ii} = 0$ for $1 \leq i \leq n$. The symmetric matrix $(p_{ij})_{1\leq i,j\leq n}$ is called the *spectrum* of π (or $\mathcal{C}$).

It follows also from the fact that $\mathcal{I}_{\mathcal{C}_i} = (\mathbf{u}_{ij})$ that Y is a *simple* primitive multiple curve, i.e. the ideal sheaf of C in $Y = C_n$ is isomorphic to $\mathcal{O}_{C_{n-1}}$. Conversely, we show in theorem 4.7.1 that if Y is a simple primitive multiple curve, then there exists a fragmented deformation of Y.

We give in 4.4 and 4.5 a way to construct fragmented deformations by induction on n. This is used later to prove statements on fragmented deformations by induction on n.

1.6. *n-stars and structure of fragmented deformations* – An *n-star* of (S, P) is a gluing $\boldsymbol{\mathcal{S}}$ of n copies $S_1, \ldots, S_n$ of S at P, together with a morphism $\tau : \boldsymbol{\mathcal{S}} \to S$ which is an identity on each S_i. All the n-stars have the same underlying Zariski topological space $S(n)$.

An n-star is called *oblate* if some neighborhood of P can be embedded in a smooth surface. This is the case if and only $\tau^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t])/(t^n)$.

Oblate n-stars are analogous to fragmented deformations but simpler. We provide a way to build oblate n-stars by induction on n.

Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of $Y = C_n$. We associate to it an oblate n-star $\boldsymbol{\mathcal{S}}$ of S. Let $\mathcal{C}^{top}$ be the Zariski topological space of $\mathcal{C}$. We have an obvious continuous map $\widetilde{\pi} : \mathcal{C}^{top} \to S(n)$. For every open subset U of $S(n)$, $\mathcal{O}_{\boldsymbol{\mathcal{S}}}(U)$ is the set of $(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}}(\widetilde{\pi}^{-1}(U))$ such that $\alpha_i \in \mathcal{O}_{S_i}(U \cap S_i)$ for $1 \leq i \leq n$. We obtain also a canonical morphism $\mathbf{\Pi} : \mathcal{C} \longrightarrow \boldsymbol{\mathcal{S}}$. We prove (theorem 5.6.2) that $\mathbf{\Pi}$ is flat. Hence it is a flat family of smooth curves, with $\mathbf{\Pi}^{-1}(P) = C$. The converse is also true, i.e. starting from an oblate n-star of S and a flat family of smooth curves parametrized by it, we obtain a fragmented deformation of a multiple primitive curve of multiplicity n.

1.7. *Fragmented deformations of double curves* – Let $Y = C_2$ be a primitive double curve, C its associated smooth curve, $\pi : \mathcal{C} \to S$ a fragmented deformation of Y, of spectrum $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$, and $\mathcal{C}_1$, $\mathcal{C}_2$ the irreducible components of $\mathcal{C}$. For $i = 1, 2$, $q > 0$, let C_i^q be the infinitesimal neighborhood of order q of C in $\mathcal{C}_i$ (defined by the ideal sheaf (π_i^q)). It is a primitive multiple curve of multiplicity q.

It follows from 4.3.5 that C_1^p and C_2^p are isomorphic, and C_1^{p+1}, C_2^{p+1} are two extensions of C_1^p in primitive multiple curves of multiplicity $p+1$. According to [4] these extensions are parametrized by an affine space with associated vector space $H^1(C, T_C)$ (where T_C is the tangent bundle of C). Let $w \in H^1(C, T_C)$ be the vector from C_1^{p+1} to C_2^{p+1}.

Similarly, the primitive double curves with associated smooth curve C such that $\mathcal{I}_C \simeq \mathcal{O}_C$ are parametrized by $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ (cf. [2], [4]).

We prove in theorem 6.0.5 that the point of $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ corresponding to C_2 is $\mathbb{C}w$.

1.8. *Notation:* Let X be an algebraic variety and $Y \subset X$ a closed subvariety. We will denote by $\mathcal{I}_{Y,X}$ (or $\mathcal{I}_Y$ if there is no risk of confusion) the ideal sheaf of Y in X.

2. Preliminaries

2.1. Local embeddings in smooth varieties

2.1.1. Proposition: *Let X be an algebraic variety, x a closed point of X and n a positive integer. Then the three following properties are equivalent:*

(i) *There exist a neighborhood U of x and an embedding $U \subset Z$ in a smooth variety of dimension n.*
(ii) *The $\mathcal{O}_{X,x}$-module $m_{X,x}$ (maximal ideal of x) can be generated by n elements.*
(iii) *We have $\dim_{\mathbb{C}}(m_{X,x}/m_{X,x}^2) \leq n$.*

Proof. It is obvious that (i) implies (ii), and (ii),(iii) are equivalent according to Nakayama's lemma. It remains to prove that (iii) implies (i).

Suppose that (iii) is true. There exist an integer N and an embedding $X \subset \mathbb{P}_N$. Let $\mathcal{I}_X$ be the ideal sheaf of X in $\mathbb{P}_N$. Let p be the biggest integer such that there exists $f_1, \cdots, f_p \in \mathcal{I}_{X,x}$ whose images in the $\mathbb{C}$-vector space $m_{\mathbb{P}_N,x}/m_{\mathbb{P}_N,x}^2$ are linearly independent. Then we have

$$\mathcal{I}_{X,x} \subset (f_1, \cdots, f_p) + m_{\mathbb{P}_N,x}^2.$$

In fact, let $f \in \mathcal{I}_{X,x}$. Since p is maximal, the image of f in $m_{\mathbb{P}_N,x}/m_{\mathbb{P}_N,x}^2$ is a linear combination of those of $f_1, \cdots, f_n$. Hence we can write

$$f = \sum_{i=1}^{p} \lambda_i f_i + g, \quad \text{with} \quad \lambda_i \in \mathbb{C},\ g \in m_{\mathbb{P}_N,x}^2 ,$$

and our assertion is proved. It follows that we have a surjective morphism

$$\alpha : \mathcal{O}_{X,x}/m_{X,x}^2 \longrightarrow \mathcal{O}_{\mathbb{P}_N,x}/\big((f_1, \cdots, f_p) + m_{\mathbb{P}_N,x}^2\big) .$$

We have

$$\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/m_{X,x}^2) \leq n+1, \quad \dim_{\mathbb{C}} \big(\mathcal{O}_{\mathbb{P}_N,x}/\big((f_1, \cdots, f_p) + m_{\mathbb{P}_N,x}^2\big)\big) = N-p+1 .$$

Hence $N-p+1 \leq n+1$, i.e. $p \geq N-n$. We can take for Z a neighborhood of x in the subvariety of $\mathbb{P}_N$ defined by $f_1, \cdots, f_{N-n}$, which is smooth at x. □

2.2. Flat families of coherent sheaves

Let (S, P) be the germ of a smooth curve and $t \in \mathcal{O}_{S,P}$ a generator of the maximal ideal. Let $\pi : X \to S$ be a flat morphism. If $\mathcal{E}$ is a coherent sheaf on X, $\mathcal{E}$ is flat on S at $x \in \pi^{-1}(P)$ if and only if the multiplication by $t : \mathcal{E}_x \to \mathcal{E}_x$ is injective. In particular the multiplication by $t : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is injective.

2.2.1. Lemma: *Let $\mathcal{E}$ be a coherent sheaf on X flat on S. Then, for every open subset U of X, the restriction $\mathcal{E}(U) \to \mathcal{E}(U\backslash\pi^{-1}(P))$ is injective.*

Proof. Let $s \in \mathcal{E}(U)$ whose restriction to $U\backslash\pi^{-1}(P)$ vanishes. We must show that $s = 0$. By covering U with smaller open subsets we can suppose that U is affine: $U = \operatorname{spec}(A)$. Hence $U\backslash\pi^{-1}(P) = \operatorname{spec}(A_t)$. Let $M = \mathcal{E}(U)$, it is an A-module. We have $\mathcal{E}_{|U} = \widetilde{M}$ and $\mathcal{E}(U\backslash\pi^{-1}(P)) = M_t$. Hence if the restriction of s to $U\backslash\pi^{-1}(P)$ vanishes, there exists an integer

$n > 0$ such that $t^n s = 0$. Since the multiplication by t is injective (because $\mathcal{E}$ is flat on S), we have $s = 0$. $\square$

Let $\mathcal{E}$ be a coherent sheaf on X flat on S. Let $\mathcal{F} \subset \mathcal{E}_{|X\backslash\pi^{-1}(P)}$ be a subsheaf. For every open subset U of X we denote by $\overline{\mathcal{F}}(U)$ the subset of $\mathcal{F}(U\backslash\pi^{-1}(P))$ of elements that can be extended to sections of $\mathcal{E}$ on U. If $V \subset U$ is an open subset, the restriction $\mathcal{F}(U\backslash\pi^{-1}(P)) \to \mathcal{F}(V\backslash\pi^{-1}(P))$ induces a morphism $\overline{\mathcal{F}}(U) \to \overline{\mathcal{F}}(V)$.

2.2.2. Proposition: *$\overline{\mathcal{F}}$ is a subsheaf of $\mathcal{E}$, and $\mathcal{E}/\overline{\mathcal{F}}$ is flat on S.*

Proof. To prove the first assertion, we must show that if U is an open subset of X and $(U_i)_{i\in I}$ is an open cover of U, then

(i) If $s \in \overline{\mathcal{F}}(U)$ is such that for every i we have $s_{|U_i} = 0$, then $s = 0$.
(ii) For every $i \in I$ let $s_i \in \overline{\mathcal{F}}(U_i)$. Then if for all i, j we have $s_{i|U_{ij}} = s_{j|U_{ij}}$, then there exists $s \in \overline{\mathcal{F}}(U)$ such that for every $i \in I$ we have $s_{|U_i} = s_i$.

This follows easily from lemma 2.2.1.

Now we prove that $\mathcal{E}/\overline{\mathcal{F}}$ is flat on S. Let $x \in \pi^{-1}(P)$ and $u \in (\mathcal{E}/\overline{\mathcal{F}})_x$ such that $tu = 0$. We must show that $u = 0$. Let $v \in \mathcal{E}_x$ over u. Then we have $tv \in \overline{\mathcal{F}}_x$. Let U be a neighborhood of x such that tv comes from $w \in \overline{\mathcal{F}}(U)$. This means that $w_{|U\backslash\pi^{-1}(P)} \in \mathcal{F}(U\backslash\pi^{-1}(P))$. Since t is invertible on $U\backslash\pi^{-1}(P)$ we can write $w = tw'$, with $w' \in \mathcal{F}(U\backslash\pi^{-1}(P))$. We have then $w' = v$ on $U\backslash\pi^{-1}(P)$. Hence $v \in \overline{\mathcal{F}}_x$ and $u = 0$. $\square$

2.3. Primitive multiple curves

(cf. [1], [2], [3], [4], [6], [7], [8], [10]).

Let C be a smooth connected projective curve. A *multiple curve with support* C is a Cohen-Macaulay scheme Y such that $Y_{red} = C$.

Let n be the smallest integer such that $Y = C^{(n-1)}$, $C^{(k-1)}$ being the k-th infinitesimal neighborhood of C, i.e. $\mathcal{I}_{C^{(k-1)}} = \mathcal{I}_C^k$. We have a filtration $C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$ where C_i is the biggest Cohen-Macaulay subscheme contained in $Y \cap C^{(i-1)}$. We call n the *multiplicity* of Y.

We say that Y is *primitive* if, for every closed point x of C, there exists a smooth surface S, containing a neighborhood of x in Y as a locally closed subvariety. In this case, $L = \mathcal{I}_C/\mathcal{I}_{C_2}$ is a line bundle on C and we have $\mathcal{I}_{C_j} = \mathcal{I}_X^j$, $\mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}} = L^j$ for $1 \leq j < n$. We call L the line bundle on C *associated* to Y. Let $P \in C$. Then there exist elements y, t of $m_{S,P}$ (the maximal ideal of $\mathcal{O}_{S,P}$) whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1 \leq i < n$ we have $\mathcal{I}_{C_i,P} = (y^i)$.

The simplest case is when Y is contained in a smooth surface S. Suppose that Y has multiplicity n. Let $P \in C$ and $f \in \mathcal{O}_{S,P}$ a local equation of C. Then we have $\mathcal{I}_{C_i,P} = (f^i)$ for $1 < j \leq n$, in particular $I_{Y,P} = (f^n)$, and $L = \mathcal{O}_C(-C)$.

We will write $\mathcal{O}_n = \mathcal{O}_{C_n}$ and we will see $\mathcal{O}_i$ as a coherent sheaf on C_n with schematic support C_i if $1 \leq i < n$.

If $\mathcal{E}$ is a coherent sheaf on Y one defines its *generalized rank* $R(\mathcal{E})$ and *generalized degree* $\mathrm{Deg}(\mathcal{E})$ (cf. [6], 3-): take any filtration of $\mathcal{E}$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

by subsheaves such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is concentrated on C for $1 \leq i \leq n$, then

$$R(\mathcal{E}) = \sum_{i=1}^{n} \mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1}) \quad \text{and} \quad \mathrm{Deg}(\mathcal{E}) = \sum_{i=1}^{n} \deg(\mathcal{E}_i/\mathcal{E}_{i-1}).$$

Let $\mathcal{O}_Y(1)$ be a very ample line bundle on Y. Then the Hilbert polynomial of $\mathcal{E}$ is

$$P_{\mathcal{E}}(m) \;=\; R(\mathcal{E})\deg(\mathcal{O}_C(1))m + \mathrm{Deg}(\mathcal{E}) + R(\mathcal{E})(1-g)$$

(where g is the genus of C).

We deduce from proposition 2.1.1:

2.3.1. Proposition: *Let Y be a multiple curve with support C. Then Y is a primitive multiple curve if and only if $\mathcal{I}_C/\mathcal{I}_C^2$ is zero, or a line bundle on C.*

2.3.2. *Parametrization of double curves* - In the case of double curves, D. Bayer and D. Eisenbud have obtained in [2] the following classification: if Y is of multiplicity 2, we have an exact sequence of vector bundles on C

$$0 \longrightarrow L \longrightarrow \Omega_{Y|C} \longrightarrow \omega_C \longrightarrow 0$$

which is split if and only if Y is the *trivial curve*, i.e. the second infinitesimal neighborhood of C, embedded by the zero section in the dual bundle L^*, seen as a surface. If Y is not trivial, it is completely determined by the line of $\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C, L)$ induced by the preceding exact sequence. The non trivial primitive curves of multiplicity 2 and of associated line bundle L are therefore parametrized by the projective space $\mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C, L))$.

2.4. Simple primitive multiple curves

Let C be a smooth projective irreducible curve, $n \geq 2$ an integer and C_n a primitive multiple curve of multiplicity n and associated reduced curve C. Then the ideal sheaf $\mathcal{I}_C$ of C in C_n is a line bundle on C_{n-1}.

We say that C_n is *simple* if $\mathcal{I}_C \simeq \mathcal{O}_{n-1}$.

In this case the line bundle on C associated to C_n is $\mathcal{O}_C$. The following result is proved in [8] (théorème 1.2.1):

2.4.1. Theorem: *Suppose that C_n is simple. Then there exists a flat family of smooth projective curves $\tau : \mathcal{C} \to \mathbb{C}$ such that $\tau^{-1}(0) \simeq C$ and that C_n is isomorphic to the n-th infinitesimal neighborhood of C in $\mathcal{C}$.*

3. Reducible reduced deformations of primitive multiples curves

3.1. Connected Components

Let (S,P) be the germ of a smooth curve and $t\in\mathcal{O}_{S,P}$ a generator of the maximal ideal. Let $n>0$ be an integer and $Y=C_n$ a projective primitive multiple curve of multiplicity n.

Let $k>0$ be an integer. Let $\pi:\mathcal{C}\to S$ be a flat morphism, where $\mathcal{C}$ is a reduced algebraic variety, such that

– For every closed point $s\in S$ such that $s\neq P$, the fiber $\mathcal{C}_s$ has k irreducible components, which are smooth and transverse, and any three of these components have no common point.
– The fiber $\mathcal{C}_P$ is isomorphic to C_n.

It is easy to see that the irreducible components of $\mathcal{C}$ are reduced surfaces.

Let Z be the open subset of $\mathcal{C}\backslash\mathcal{C}_P$ of points z belonging to only one irreducible component of $\mathcal{C}_{\pi(z)}$. Then the restriction of $\pi:Z\to S\backslash\{P\}$ is a smooth morphism. For every $s\in S\backslash\{P\}$, let $\mathcal{C}'_s=\mathcal{C}_s\cap Z$. It is the open subset of smooth points of $\mathcal{C}_s$.

Let $z\in Z$ and $s=\pi(z)$. There exist a neighborhood (for the Euclidean topology) U of s, isomorphic to $\mathbb{C}$, and a neighborhood V of z such that $V\simeq\mathbb{C}^2$, $\pi(V)=U$, the restriction of $\pi:V\to U$ being the projection $\mathbb{C}^2\to\mathbb{C}$ on the first factor. We deduce easily from that the following facts:

– let $s\in S\backslash\{P\}$ and C_1 an irreducible component of $\mathcal{C}_s$. Let $z_1,z_2\in C_1\cap Z$. Then there exist neighborhoods (in Z, for the Euclidean topology) U_1, U_2 of z_1, z_2 respectively, such that if $y_1\in U_1$, $y_2\in U_2$ are such that $\pi(y_1)=\pi(y_2)$, then y_1 and y_2 belong to the same irreducible component of $\mathcal{C}_{\pi(y_1)}$.
– for every continuous map $\sigma:[0,1]\to S\backslash\{P\}$ and every $z\in Z$ such that $\sigma(0)=\pi(z)$ there exists a lifting of σ, $\sigma':[0,1]\to Z$ such that $\sigma'(0)=z$. Moreover, if $\sigma'':[0,1]\to Z$ is another lifting of σ such that $\sigma''(0)=z$, then $\sigma'(1)$ and $\sigma''(1)$ are in the same irreducible component of $\mathcal{C}_{\sigma(1)}$. More generally, if we only impose that $\sigma''(0)$ is in the same irreducible component of $\mathcal{C}_{\sigma(0)}$ as z, then $\sigma'(1)$ and $\sigma''(1)$ are in the same irreducible component of $\mathcal{C}_{\sigma(1)}$.

3.1.1. Lemma: *Let $\sigma_0,\sigma_1:[0,1]\to S\backslash\{P\}$ be two continuous maps such that $\sigma_0(0)=\sigma_1(0)$, $s=\sigma_0(1)=\sigma_1(1)$. Suppose that they are homotopic. Let σ'_0, σ'_1 be liftings $[0,1]\to Z$ of σ_0, σ_1 respectively, such that $\sigma'_0(0)=\sigma'_1(0)$. Then $\sigma'_0(1)$ and $\sigma'_1(1)$ belong to the same irreducible component of $\mathcal{C}'_s$.*

Proof. Let
$$\Psi:[0,1]\times[0,1]\longrightarrow S\backslash\{P\}$$
be an homotopy:
$$\Psi(0,t)=\sigma_0(t),\quad \Psi(1,t)=\sigma_1(t),\quad \Psi(t,0)=\sigma_0(0),\quad \Psi(t,1)=\sigma_0(1)$$
for $0\leq t\leq 1$. For every $u\in[0,1]$ and $\epsilon>0$ let $I_{u,\epsilon}=[u-\epsilon,u+\epsilon]\cap[0,1]$. By using the local structure of $\pi_{|Z}$ for the Euclidean topology it is easy to see that for every $u\in[0,1]$, there exists

an $\epsilon > 0$ such that the restriction of Ψ

$$I_{u,\epsilon} \times [0,1] \longrightarrow S\backslash\{P\}$$

can be lifted to a morphism

$$\Psi' : I_{u,\epsilon} \times [0,1] \longrightarrow Z$$

such that $\Psi'(t,0) = \sigma'_0(0)$ for every $t \in I_{u,\epsilon}$. It follows that if $I_{u,\epsilon} = [a_{u,\epsilon}, b_{u,\epsilon}]$, then $\Psi'(a_{u,\epsilon}, 1)$ and $\Psi'(b_{u,\epsilon}, 1)$ are in the same irreducible component of $\mathcal{C}'_{\sigma_0(1)}$. We have just to cover $[0,1]$ with a finite number of intervals $I_{u,\epsilon}$ to obtain the result. □

Let $s \in S\backslash\{P\}$, $D_1, \ldots, D_k$ be the irreducible components of $\mathcal{C}'_s$ and $x_i \in D_i$ for $1 \leq i \leq k$. Let σ be a loop of $S\backslash\{P\}$ with origin s, defining a generator of $\pi_1(S\backslash\{P\})$. Let i be an integer such that $1 \leq i \leq k$. The liftings $\sigma' : [0,1] \to Z$ of σ such that $\sigma'(0) = x_i$ end up at a component D_j which does not depend on x_i. Hence we can write

$$j \;= \alpha_{\mathcal{C}}(i).$$

3.1.2. Lemma: *$\alpha_{\mathcal{C}}$ is a permutation of $\{1, \ldots, k\}$.*

Proof. Suppose that $i \neq j$ and $\alpha_{\mathcal{C}}(i) = \alpha_{\mathcal{C}}(j)$. By inverting the paths we find liftings of paths from $D_{\alpha_{\mathcal{C}}(i)}$ to D_i and D_j. This contradicts lemma 3.1.1. □

Let $p > 0$ be an integer such that $\alpha_{\mathcal{C}}^p = I_{\{1,\cdots,k\}}$. Let t be a generator of the maximal ideal of $\mathcal{O}_{S,P}$, K the field of rational functions on S and $K' = K(t^{1/p})$. Let S' be the germ of the curve corresponding to K', $\theta : S' \to S$ canonical the morphism and P' the unique point of $\theta^{-1}(P)$. Let $\mathcal{D} = \theta^*(\mathcal{C})$. We have therefore a cartesian diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\rho"] \arrow[d, "\Theta"] & S' \arrow[d, "\theta"] \\ \mathcal{C} \arrow[r, "\pi"] & S \end{tikzcd}$$

where ρ is flat, and for every $s' \in S'$, Θ induces an isomorphism $\mathcal{D}_{s'} \simeq \mathcal{C}_{\theta(s')}$. We have

$$\alpha_{\mathcal{D}} \;=\; I_{\{1,\ldots,k\}}\ .$$

Let $Z' \subset \mathcal{D}$ be the complement of the union of $\rho^{-1}(P')$ and of the singular points of the curves $\mathcal{D}_{s'}$, $s' \neq P'$ (hence $Z' = \Theta^{-1}(Z)$).

3.1.3. Proposition: *The open subset Z' has exactly k irreducible components $Z'_1, \ldots, Z'_k$. Let $\overline{Z'_1}, \ldots, \overline{Z'_k}$ be their closures in $\mathcal{D}$. Then for every $s' \in S'\backslash\{P'\}$, the $Z'_i \cap \mathcal{D}_{s'}$, $1 \leq i \leq k$, are the irreducible components of $\mathcal{D}_{s'}$ minus the intersection points with the other components, and the $\overline{Z'_i} \cap \mathcal{D}_{s'}$ are the irreducible components of $\mathcal{D}_{s'}$.*

3.1.4. Definition: *Let $k > 0$ be an integer. A* reducible deformation of length k of C_n *is a flat morphism $\pi : \mathcal{C} \to S$, where $\mathcal{C}$ is a reduced algebraic variety, such that*

- *For every closed point $s \in S$, $s \neq P$, the fiber $\mathcal{C}_s$ has k irreducible components, which are smooth and transverse, and any three of these components have no common point.*

– *The fiber* $\mathcal{C}_P$ *is isomorphic to* C_n.
– *We have* $\alpha_{\mathcal{C}} = I_{\{1,\dots,k\}}$.

3.2. Maximal reducible deformations

Let (S,P) be the germ of a smooth curve and $t\in\mathcal{O}_{S,P}$ a generator of the maximal ideal. Let $n>0$ be an integer and $Y=C_n$ a projective primitive multiple curve of multiplicity n, with underlying smooth curve C. We denote by g the genus of C and L the line bundle on C associated to C_n.

Let $\pi:\mathcal{C}\to S$ be a reducible deformation of length k of C_n. Let $Z_1,\dots,Z_k$ be the closed subvarieties of $\pi^{-1}(S\backslash\{P\})$ such that for every $s\in S\backslash\{P\}$, $Z_{1s},\dots,Z_{ks}$ are the irreducible components of $\mathcal{C}_s$ (cf. prop. 3.1.3).

For $1\leq i\leq k$, we denote by $\mathcal{J}_i$ the ideal sheaf of $Z_1\cup\cdots\cup Z_i$ in $\pi^{-1}(S\backslash\{P\})$. This sheaf is flat on $S\backslash\{P\}$, and we have
$$0=\mathcal{J}_k\subset\mathcal{J}_{k-1}\subset\cdots\subset\mathcal{J}_1\subset\mathcal{O}_{\pi^{-1}(S\backslash\{P\})}\ .$$
The quotients $\mathcal{O}_{\pi^{-1}(S\backslash\{P\})}/\mathcal{J}_1$, $\mathcal{J}_i/\mathcal{J}_{i+1}$, $1\leq i<k$, are also flat on $S\backslash\{P\}$. We obtain the filtration of sheaves on $\mathcal{C}$
$$0=\overline{\mathcal{J}_k}\subset\overline{\mathcal{J}_{k-1}}\subset\cdots\subset\overline{\mathcal{J}_1}\subset\mathcal{O}_{\mathcal{C}}\ .$$
(cf. 2.2). According to proposition 2.2.2 the quotients $\mathcal{O}_{\mathcal{C}}/\overline{\mathcal{J}_1}$ and $\overline{\mathcal{J}_i}/\overline{\mathcal{J}_{i+1}}$, $1\leq i<n$, are flat on S. We have $\mathcal{O}_{\pi^{-1}(S\backslash\{P\})}/\mathcal{J}_1=\mathcal{O}_{Z_1}$. We denote by $\mathbf{X}_i$ the closed subvariety of $\mathcal{C}$ corresponding to the ideal sheaf $\overline{\mathcal{J}_i}$.

Similarly we consider the ideal sheaf $\mathcal{J}'_i$ of $Z_{i+1}\cup\cdots\cup Z_n$ on $\pi^{-1}(S\backslash\{P\})$, the associated ideal sheaf $\overline{\mathcal{J}'_i}$ on $\mathcal{C}$ and the corresponding subvariety $\mathbf{X}'_i$.

3.2.1. Proposition: *We have* $k\leq n$.

Proof. Let $\mathcal{E}_0=\mathcal{O}_{\mathcal{C}}/\overline{\mathcal{J}_1}$ and $\mathcal{E}_i=\overline{\mathcal{J}_i}/\overline{\mathcal{J}_{i+1}}$ for $1\leq i<n$. The sheaves $\mathcal{E}_{iP}$ are not concentrated on a finite number of points. To see this we use a very ample line bundle $\mathcal{O}(1)$ on $\mathcal{C}$. The Hilbert polynomial of $\mathcal{E}_{iP}$ is the same as that of $\mathcal{E}_{is}$, $s\neq P$, hence it is not constant. So we have $R(\mathcal{E}_i)\geq 1$ ($R(\mathcal{E}_i)$ is the generalized rank of $\mathcal{E}_i$, cf. 2.3), and since

$$n\ =\ R(\mathcal{O}_{C_n})\ =\sum_{i=0}^{k}R(\mathcal{E}_{iP})\ , \tag{1}$$

we have $k\leq n$. □

3.2.2. Definition: *We say that* π *(or* $\mathcal{C}$*) is a* maximal reducible deformation *of* C_n *if* $k=n$.

3.2.3. Theorem: *Suppose that* $\mathcal{C}$ *is a maximal reducible deformation of* C_n*. Then we have, for* $1\leq i<n$
$$\overline{\mathcal{J}_i}_{,P}\ =\ \mathcal{I}_{C_i,C_n}$$
and $\mathbf{X}_i$ *is a maximal reducible deformation of* C_i.

Proof. Let $\mathcal{O}_\mathcal{C}(1)$ be a very ample line bundle on $\mathcal{C}$.

Let Q be a closed point of C. Let $z \in \mathcal{O}_{n,Q}$ be an equation of C and $x \in \mathcal{O}_{n,Q}$ over a generator of the maximal ideal of Q in $\mathcal{O}_{C,Q}$. Let $\mathbf{z}, \mathbf{x} \in \mathcal{O}_{\mathcal{C},Q}$ be over z, x respectively. The maximal ideal of $\mathcal{O}_{n,Q}$ is (x, z). The maximal ideal of $\mathcal{O}_{\mathcal{C},Q}$ is generated by $\mathbf{z}, \mathbf{x}, t$. It follows from proposition 2.1.1 that there exist a neighborhood U of Q in $\mathcal{C}$ and an embedding $j : U \to \mathbb{P}_3$. We can assume that the restriction of j to $\overline{Z_1} \cap U$ is induced by the morphism $\phi : \mathbb{C}[X, Z, T] \to \mathcal{O}_{\overline{Z_1},Q}$ of $\mathbb{C}$-algebras which associates x, z, t to X, Z, T respectively.

Since $\mathcal{C}$ is reduced, U is an open subset of a reduced hypersurface of $\mathbb{P}_3$ having n irreducible components, corresponding to $\overline{Z_1}, \ldots, \overline{Z_n}$. It is then clear that $\mathbf{X}_i$, being the smallest subscheme of $\mathcal{C}$ containing $Z_1 \backslash C, \ldots, Z_i \backslash C$, is the union in U of the first i hypersurface components.

Since $j(\overline{Z_1})$ is a hypersurface, the kernel of ϕ is a principal ideal generated by the equation F of the image of Z_1.

Recall that $\mathcal{O}_n = \mathcal{O}_{C_n} = (\mathcal{O}_\mathcal{C})_P$. We have $R(\mathcal{O}_n/\overline{\mathcal{J}_1,}_P) = 1$ according to (1). Hence there exists a nonempty open subset V of C_n such that $\big(\mathcal{O}_n/\overline{\mathcal{J}_1,}_P\big)_{|V}$ is a line bundle on $V \cap C$. It follows that the projection $\mathcal{O}_n \to \mathcal{O}_C$ vanishes on $\overline{\mathcal{J}_1,}_{P|V}$. Since $\mathcal{O}_C$ is torsion free this projection vanishes everywhere on $\overline{\mathcal{J}_1}$, i.e. $\overline{\mathcal{J}_1}_P \subset \mathcal{I}_{C,C_n}$, with equality on V.

The sheaf $\mathcal{E}_0 = \mathcal{O}_\mathcal{C}/\overline{\mathcal{J}_1}$ is the structural sheaf of $\overline{Z_1}$, and the projection $\overline{Z_1} \to S$ is a flat morphism. For every $s \in S \backslash \{P\}$, $(\overline{Z_1})_s$ is a smooth curve. The fiber $(\overline{Z_1})_P$ consists of C and a finite number of embedded points. There exist flat families of curves whose general fiber is smooth and the special fiber consists of an integral curve and some embedded points (cf. [12], III, Example 9.8.4). We will show that this cannot happen in our case, i.e. we have $\overline{\mathcal{J}_1}_P = \mathcal{I}_{C,C_n}$.

Let $\mathbf{m} = (X, Z, T) \subset \mathbb{C}[X, Z, T]$, and $\mathbf{m}_{Z_1}$ the maximal ideal of $\mathcal{O}_{\overline{Z_1},Q}$. The ideal of $(\overline{Z_1})_P$ in $\mathcal{O}_{n,Q}$ contains z^q and $x^p z$ (for suitable minimal integers $p \geq 0, q > 0$), with $p > 0$ if and only if Q is an embedded point. Hence the ideal of $\overline{Z_1}$ in $\mathcal{O}_{\mathcal{C},Q}$ contains elements of type $\mathbf{x}^p\mathbf{z} - t\alpha$, $\mathbf{z}^q - t\beta$, with $\alpha, \beta \in \mathcal{O}_{\mathcal{C},Q}$.

Let $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$ be the completion of $\mathcal{O}_{\overline{Z_1},Q}$ with respect to $\mathbf{m}_{Z_1}$ and

$$\widehat{\phi} : \mathbb{C}((X, Z, T)) \longrightarrow \widehat{\mathcal{O}_{\overline{Z_1},Q}}$$

the morphism deduced from ϕ. We can also see $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$ as the completion with respect to (X, Z, T) of $\mathcal{O}_{\overline{Z_1},Q}$ seen as a $\mathbb{C}[X, Z, T]$-module. It follows that $\ker(\widehat{\phi}) = (F)$ (cf. [9], lemma 7.15). Note that $\widehat{\phi}$ is surjective (this is why we use completions). Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}((X, Y, Z))$ be such that $\widehat{\phi}(\boldsymbol{\alpha}) = \alpha$, $\widehat{\phi}(\boldsymbol{\beta}) = \beta$. So we have

$$X^p Z - T\boldsymbol{\alpha},\ Z^q - T\boldsymbol{\beta}\ \in\ \ker(\widehat{\phi})\ .$$

Hence there exist $A, B \in \mathbb{C}((X, Z, T))$ such that $X^p Z - T\boldsymbol{\alpha} = AF$, $Z^q - T\boldsymbol{\beta} = BF$. We can write in an unique way

$$A = A_0 + TA_1,\ B = B_0 + TB_1,\ F = F_0 + TF_1,$$

with $A_0, B_0, F_0 \in \mathbb{C}((X, Z))$ and $A_1, B_1, F_1 \in \mathbb{C}((X, Z, T))$, and we have

$$A_0 F_0 = X^p Z,\ B_0 F_0 = Z^q\ .$$

Since F is not invertible, it follows that F_0 is of the form $F_0 = cZ$, with $c \in \mathbb{C}((X, Z, T))$ invertible. So we have $F = cZ + TF_1$. It follows that $z \in (t)$ in $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$. This implies that this is also true in $\mathcal{O}_{\overline{Z_1},Q}$: in fact the assertion in $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$ implies that

$$z \ \in \ \bigcap_{n \geq 0} ((t) + \mathbf{m}_{Z_1}^n)$$

in $\mathcal{O}_{\overline{Z_1},Q}$, and the latter is equal to (t) according to [14], vol. II, chap. VIII, theorem 9. Hence $z \in (t)$ in $\mathcal{O}_{\overline{Z_1},Q}$, i.e. $p = 0$ and Q is not an embedded point. So there are no embedded points. This implies that $\overline{\mathcal{J}_1}_P = \mathcal{I}_{C,C_n}$. Similarly, if I_j denotes the ideal sheaf of $\overline{Z_j}$ for $1 \leq j \leq n$, we have $I_{j,P} = \mathcal{I}_{C,C_n}$. Since the restriction of $\pi : \overline{Z_j} \to S$ is flat, the curves $\mathcal{E}_{j,s}$, $s \neq P$, have the same genus as C, and the same Hilbert polynomial with respect to $\mathcal{O}_{\mathcal{C}}(1)$.

Now we show that $\mathbf{X}_1'$ is a maximal reducible deformation of C_{n-1}. We need only to show that $\mathbf{X}_{1,P}' = C_{n-1}$. As we have seen, for $2 \leq j \leq n$, a local equation of $\overline{Z_j}$ at any point $Q \in C$ induces a generator u_j of $\mathcal{I}_{C,C_n,Q}$. Hence $u = \prod_{2 \leq j \leq n} u_j$ is a generator of $\mathcal{I}_{C_{n-1},C_n,Q}$. But $u = 0$ on $\mathbf{X}_1'$. It follows that $\mathbf{X}_{1,P}' \subset C_{n-1}$. But the Hilbert polynomial of $\mathcal{O}_{C_{n-1}}$ is the same as that of the structural sheaves of the fibers of the flat morphism $\mathbf{X}_1' \to S$ over $s \neq P$, hence the same as $\mathcal{O}_{\mathbf{X}_{1,P}'}$. Hence $\mathbf{X}_{1,P}' = C_{n-1}$.

The theorem 3.2.3 is then easily proved by induction on n. □

3.2.4. Corollary: *Let $s \in S \backslash \{P\}$ and D_1, D_2 be two irreducible components of $\mathcal{C}_s$. Then D_1 is of genus g and $D_1 \cap D_2$ consists of $-\deg(L)$ points.*

Proof. According to theorem 3.2.3, there exists a flat family of smooth curves $\mathbf{C}$ parametrized by S such that $\mathbf{C}_P = C$ and $\mathbf{C}_s = D_1$. So the genus of D_1 is equal to that of C.

Let us prove the second assertion. Again according to theorem 3.2.3 we can suppose that $n = 2$. We have then $\chi(\mathcal{C}_s) = \chi(C_2) = 2\chi(C) + \deg(L)$. Let $x_1, \ldots, x_N$ be the intersection points of D_1 and D_2. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{D_2}(-x_1 - \cdots - x_N) \longrightarrow \mathcal{O}_{\mathcal{C}_s} \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0.$$

Whence $\chi(\mathcal{O}_{\mathcal{C}_s}) = \chi(D_1) + \chi(D_2) - N = 2\chi(\mathcal{O}_C) - N$ (according to the first assertion). Whence $N = -\deg(L)$. □

3.2.5. It follows from the previous results that if $\pi : \mathcal{C} \to S$ is a maximal reducible deformation of C_n, then we have

(i) $\deg(L) \leq 0$.
(ii) $\mathcal{C}$ has exactly n irreducible components $\mathcal{C}_1 \ldots, \mathcal{C}_n$.
(iii) For $1 \leq i \leq n$, the restriction of π, $\pi_i : \mathcal{C}_i \to S$ is a flat morphism , and $\pi_i^{-1}(P) = C$.
(iv) For every nonempty subset $I \subset \{1, \ldots, n\}$, let $\mathcal{C}_I$ be the union of the $\mathcal{C}_i$ such that $i \in I$, and m the number of elements of I. Then the restriction of π, $\pi_I : \mathcal{C}_I \to S$ is a maximal reducible deformation of C_m.

The following is immediate, and shows that we need only to consider maximal reducible deformations parametrized by a neighborhood of 0 in $\mathbb{C}$:

3.2.6. Proposition: *Let $t \in \mathcal{O}_S(P)$ be a generator of the maximal ideal, and $\pi : \mathcal{C} \to S$ a maximal reducible deformation of C_n. Let $S' \subset S$ be an open neighborhood of P where t is defined and $\mathcal{C}' = \pi^{-1}(U)$, $V = t(U)$. Then $\pi' = t \circ \pi : \mathcal{C}' \to V$ is a maximal reducible deformation of C_n.*

4. Fragmented deformations of primitive multiple curves

The fragmented deformations of primitive multiple curves are particular cases of reducible deformations.

In this chapter (S, P) denotes the germ of a smooth curve. Let $t \in \mathcal{O}_{S,P}$ be a generator of the maximal ideal of P. We can suppose that t is defined on the whole of S, and that the ideal sheaf of P in S is generated by t.

4.1. Fragmented deformations and gluing

Let $n > 0$ be an integer and $Y = C_n$ a projective primitive multiple curve of multiplicity n.

4.1.1. Definition: *Let $k > 0$ be an integer. A* general fragmented deformation of length k *of C_n is a flat morphism $\pi : \mathcal{C} \to S$ such that for every point $s \neq P$ of S, the fiber $\mathcal{C}_s$ is a disjoint union of k projective smooth irreducible curves, and such that $\mathcal{C}_P$ is isomorphic to C_n.*

We have then $k \leq n$. If $k = n$ we say that π (or $\mathcal{C}$) is a *general maximal fragmented deformation* of C_n. We suppose in the sequel that it is the case.

The line bundle on C associated to C_n is $\mathcal{O}_C$ (by proposition 3.2.4).

Let $p > 0$ be an integer. Let K be the field of rational functions on S and $K' = K(t^{1/p})$. Let S' be the germ of curve corresponding to K', $\theta : S' \to S$ the canonical morphism and P' the unique point of $\theta^{-1}(P)$. Let $\mathcal{D} = \theta^*(\mathcal{C})$. So we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\rho} & S' \\ \downarrow{\scriptstyle\Theta} & & \downarrow{\scriptstyle\theta} \\ \mathcal{C} & \xrightarrow{\pi} & S \end{array}$$

where ρ is flat, and for every $s' \in S'$, Θ induces an isomorphism $\mathcal{D}_{s'} \simeq \mathcal{C}_{\theta(s')}$.

4.1.2. Proposition: *For a suitable choice of p, $\mathcal{D}$ has exactly n irreducible components $\mathcal{D}_1, \ldots, \mathcal{D}_n$, and for every point $s \neq P'$ of S', $\mathcal{D}_{1s}, \ldots, \mathcal{D}_{ns}$ are the irreducible components of $\mathcal{D}_s$, for $1 \leq i \leq n$ the restriction of ρ: $\mathcal{D}_{is} \to S'$ is flat, and $\mathcal{D}_{P'} = C_n$.*

(See proposition 3.1.3)

4.1.3. Definition: *A* fragmented deformation *of C_n is a general maximal fragmented deformation of length n of C_n having n irreducible components.*

We suppose in the sequel that $\mathcal{C}$ is a fragmented deformation of C_n, union of n irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_n$.

4.1.4. Proposition: *Let $I \subset \{1, \ldots, n\}$ be a nonempty subset having m elements. Let $\mathcal{C}_I = \cup_{i \in I} \mathcal{C}_i$. Then the restriction of π, $\mathcal{C}_I \to S$, is flat, and the fiber $\mathcal{C}_{IP}$ is canonically isomorphic to C_m.*

(See 3.2.5)

In particular there exists a filtration of ideal sheaves

$$0 \subset \mathcal{I}_1 \subset \cdots \subset \mathcal{I}_{n-1} \subset \mathcal{O}_\mathcal{C}$$

such that for $1 \leq i < n$ and $s \in S \backslash \{P\}$, $\mathcal{I}_{is}$ is the ideal sheaf of $\cup_{j=i}^n \mathcal{C}_{js}$, and that $\mathcal{I}_{iP}$ is that of C_{n-i}.

4.1.5. Definition: *For $1 \leq i \leq n$, let $\pi_i : \mathcal{C}_i \to S$ be a flat family of smooth projective irreducible curves, with a fixed isomorphism $\pi_i^{-1}(P) \simeq C$. A* gluing *of $\mathcal{C}_1, \cdots, \mathcal{C}_n$* along *$C$ is an algebraic variety $\mathcal{D}$ such that*

- *for $1 \leq i \leq n$, $\mathcal{C}_i$ is isomorphic to a closed subvariety of $\mathcal{D}$, also denoted by $\mathcal{C}_i$, and $\mathcal{D}$ is the union of these subvarieties.*
- *$\coprod_{1 \leq i \leq n} (\mathcal{C}_i \backslash C)$ is an open subset of $\mathcal{D}$.*
- *There exists a morphism $\pi : \mathcal{D} \to S$ inducing π_i on $\mathcal{C}_i$, for $1 \leq i \leq n$.*
- *The subvarieties $C = \pi_i^{-1}(P)$ of $\mathcal{C}_i$ coincide in $\mathcal{D}$.*

For example the previous fragmented deformation $\mathcal{C}$ of C_n is a gluing of $\mathcal{C}_1, \cdots, \mathcal{C}_n$ along C.

All the gluings of $\mathcal{C}_1, \cdots, \mathcal{C}_n$ along C have the same underlying Zariski topological space.

Let $\boldsymbol{\mathcal{A}}$ be the *initial gluing* of the $\mathcal{C}_i$ along C. It is an algebraic variety whose points are the same as those of $\mathcal{C}$, i.e.

$$(\coprod_{i=1}^n \mathcal{C}_i)/\sim \quad ,$$

where $\sim$ is the equivalence relation: if $x \in \mathcal{C}_i$ and $y \in \mathcal{C}_j$, $x \sim y$ if and only if $x = y$, or if $x \in \mathcal{C}_{iP} \simeq C$, $y \in \mathcal{C}_{jP} \simeq C$ and $x = y$ in C. The structural sheaf is defined by: for every open subset U of $\boldsymbol{\mathcal{A}}$

$$\mathcal{O}_{\boldsymbol{\mathcal{A}}}(U) \;=\; \{(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}_1}(U \cap \mathcal{C}_1) \times \cdots \times \mathcal{O}_{\mathcal{C}_n}(U \cap \mathcal{C}_n); \alpha_{1|C} = \cdots = \alpha_{n|C}\}.$$

For every gluing $\mathcal{D}$ of $\mathcal{C}_1, \cdots, \mathcal{C}_n$, we have an obvious dominant morphism $\boldsymbol{\mathcal{A}} \to \mathcal{D}$. If follows that the sheaf of rings $\mathcal{O}_\mathcal{D}$ can be seen as a subsheaf of $\mathcal{O}_{\boldsymbol{\mathcal{A}}}$.

The fiber $D = \boldsymbol{\mathcal{A}}_P$ is not a primitive multiple curve (if $n > 2$): if $\mathcal{I}_{C,D}$ denotes the ideal sheaf of C in D we have $\mathcal{I}_{C,D}^2 = 0$, and $\mathcal{I}_{C,D} \simeq \mathcal{O}_C \otimes \mathbb{C}^{n-1}$.

4.1.6. Proposition: *Let $\mathcal{D}$ be a gluing of $\mathcal{C}_1, \cdots, \mathcal{C}_n$. Then $\pi^{-1}(P)$ is a primitive multiple curve if and only if for every closed point x of C, there exists a neighborhood of x in $\mathcal{D}$ that can be embedded in a smooth variety of dimension 3.*

Proof. Suppose that $\pi^{-1}(P)$ is a primitive multiple curve. Then $\mathcal{I}_C/(\mathcal{I}_C^2+(\pi))$ is a principal module at x: suppose that the image of $u \in m_{\mathcal{D},x}$ is a generator. The module $m_{\mathcal{D},x}/\mathcal{I}_C$ is also principal (since it is the maximal ideal of x in C): suppose that the image of $v \in m_{\mathcal{D},x}$ is a generator. Then the images of u, v, π generate $m_{\mathcal{D},x}/m_{\mathcal{D},x}^2$, so according to proposition 2.1.1, we can locally embed $\mathcal{D}$ in a smooth variety of dimension 3.

Conversely, suppose that a neighborhood of $x \in C$ in $\mathcal{D}$ is embedded in a smooth variety Z of dimension 3. The proof of the fact that $\pi^{-1}(P)$ is Cohen-Macaulay is similar to that of theorem 3.2.3. We can suppose that π is defined on Z. We have $\pi_{|\mathcal{C}_1} = \pi_1 \notin m_{\mathcal{C}_1,x}^2$, so $\pi \notin m_{Z,x}^2$. It follows that the surface of Z defined by π is smooth at x, and that we can locally embed $\pi^{-1}(P)$ in a smooth surface. Hence $\pi^{-1}(P)$ is a primitive multiple curve. $\square$

4.2. Fragmented deformations of length 2

Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of C_2. So $\mathcal{C}$ has two irreducible components $\mathcal{C}_1, \mathcal{C}_2$. Let $\mathcal{A}$ be the initial gluing of $\mathcal{C}_1$ and $\mathcal{C}_2$ along C. For every open subset U of $\mathcal{C}$, U is also an open subset of $\mathcal{A}$ and $\mathcal{O}_{\mathcal{C}}(U)$ is a sub-algebra of $\mathcal{O}_{\mathcal{A}}(U)$. For $i = 1, 2$, let $\pi_i : \mathcal{C}_i \to S$ be the restriction of π. We will also denote $t \circ \pi$ by π, and $t \circ \pi_i$ by π_i. So we have $\pi = (\pi_1, \pi_2) \in \mathcal{O}_{\mathcal{C}}(\mathcal{C})$.

Let $\mathcal{I}_C$ be the ideal sheaf of C in $\mathcal{C}$. Since $C_2 = \pi^{-1}(P)$ we have $\mathcal{I}_C^2 \subset \langle(\pi_1, \pi_2)\rangle$.

Let $m > 0$ be an integer, $x \in C$, $\alpha_1 \in \mathcal{O}_{\mathcal{C}_1,x}$, $\alpha_2 \in \mathcal{O}_{\mathcal{C}_2,x}$. We denote by $[\alpha_1]_m$ (resp. $[\alpha_2]_m$) the image of α_1 (resp. α_2) in $\mathcal{O}_{\mathcal{C}_1,x}/(\pi_1^m)$ (resp. $\mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^m)$).

4.2.1. Proposition: 1 – *There exists an unique integer $p > 0$ such that $\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle$ is generated by the image of $(\pi_1^p, 0)$.*

2 – *The image of $(0, \pi_2^p)$ generates $\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle$.*

3 – *For every $x \in C$, $\alpha \in \mathcal{O}_{\mathcal{C}_1,x}$ and $\beta \in \mathcal{O}_{\mathcal{C}_2,x}$, we have $(\pi_1^p\alpha, 0) \in \mathcal{O}_{\mathcal{C},x}$ and $(0, \pi_2^p\beta) \in \mathcal{O}_{\mathcal{C},x}$.*

Proof. Let $x \in C$ and $u = (\pi_1\alpha, \pi_2\beta)$ whose image is a generator of $\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle$ at x ($\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle$ is a locally free sheaf of rank 1 of $\mathcal{O}_C$-modules). Let $\beta_0 \in \mathcal{O}_{\mathcal{C}_1,x}$ be such that $(\beta_0, \beta) \in \mathcal{O}_{\mathcal{C},x}$. Then the image of

$$u - (\pi_1, \pi_2)(\beta_0, \beta) = (\pi_1(\alpha - \beta_0), 0)$$

is also a generator of $\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle$ at x. We can write it $(\pi_1^p\lambda, 0)$, where λ is not a multiple of π_1.

Now we show that p is the smallest integer q such that $(\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle)_x$ contains the image of an element of the form $(\pi_1^q\mu, 0)$, with μ not divisible by π_1. We can write

$$(\pi_1^q\mu, 0) \ = \ (u_1, u_2)(\pi_1^p\lambda, 0) + (v_1, v_2)(\pi_1, \pi_2)$$

with $(u_1, u_2), (v_1, v_2) \in \mathcal{O}_{\mathcal{C},x}$. So we have $v_2 = 0$, hence $(v_1, v_2) \in \mathcal{I}_{C,x}$. So we can write (v_1, v_2) as the sum of a multiple of $(\pi_1^p\lambda, 0)$ and a multiple of (π_1, π_2). Finally we obtain $(\pi_1^q\mu, 0)$ as

$$(\pi_1^q\mu, 0) \ = \ (u_{12}, u_{22})(\pi_1^p\lambda, 0) + (v_{11}, 0)(\pi_1, \pi_2)^2.$$

In the same way we see that $(\pi_1^q\mu, 0)$ can be written as

$$(\pi_1^q\mu, 0) \ = \ (u_{1p}, u_{2p})(\pi_1^p\lambda, 0) + (v_{1p}, 0)(\pi_1, \pi_2)^p,$$

which implies immediately that $q \geq p$.

It follows that p does not depend on x and that $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ is a subsheaf of $\langle(\pi_1^p,0)\rangle/\langle(\pi_1^{p+1},0)\rangle\simeq\mathcal{O}_C$. Since $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ is of degree 0 by (by corollary 3.2.4) it follows that $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle\simeq\langle(\pi_1^p,0)\rangle/\langle(\pi_1^{p+1},0)\rangle$, from which we deduce assertion 1- of proposition 4.2.1. The second assertion comes from the fact that $(0,\pi_2^p)=\pi^p-(\pi_1^p,0)$.

To prove the third, we use the fact that there exists $\alpha'\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $(\alpha,\alpha')\in\mathcal{O}_{\mathcal{C},x}$ (because $\mathcal{C}_1\subset\mathcal{C}$). Hence $(\pi_1^p,0)(\alpha,\alpha')=(\pi_1^p\alpha,0)\in\mathcal{O}_{\mathcal{C},x}$. Similarly, we obtain that $(0,\pi_2^p\beta)\in\mathcal{O}_{\mathcal{C},x}$. $\square$

According to the proof of proposition 4.2.1, for every $x\in C$, p is the smallest integer q such that there exists an element of $\mathcal{O}_{\mathcal{C},x}$ of the form $(\pi_1^q\alpha,0)$ (resp. $(0,\pi_2^q\alpha)$), with $\alpha\in\mathcal{O}_{\mathcal{C}_1,x}$ (resp. $\alpha\in\mathcal{O}_{\mathcal{C}_2,x}$) not vanishing on C.

Let $x\in C$ and $\alpha_1\in\mathcal{O}_{\mathcal{C}_1,x}$. Since $\mathcal{C}_1\subset\mathcal{C}$ there exists $\alpha_2\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $(\alpha_1,\alpha_2)\in\mathcal{O}_{\mathcal{C},x}$. Let $\alpha_2'\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $(\alpha_1,\alpha_2')\in\mathcal{O}_{\mathcal{C},x}$. We have then $(0,\alpha_2-\alpha_2')\in\mathcal{O}_{\mathcal{C},x}$. So there exists $\alpha\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $\alpha_2-\alpha_2'=\pi_2^p\alpha$. It follows that the image of α_2 in $\mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^p)$ is uniquely determined. Hence we have:

4.2.2. Proposition: *There exists a canonical isomorphism*

$$\Phi:\mathcal{C}_1^{(p)}\longrightarrow\mathcal{C}_2^{(p)}$$

between the infinitesimal neighborhoods of order p of $\mathcal{C}_1$ and $\mathcal{C}_2$ (i.e. $\mathcal{O}_{\mathcal{C}_i^{(p)}}=\mathcal{O}_{\mathcal{C}_i}/(\pi_i^p)$), such that for every $x\in C$, $\alpha_1\in\mathcal{O}_{\mathcal{C}_1,x}$ and $\alpha_2\in\mathcal{O}_{\mathcal{C}_2,x}$, we have $(\alpha_1,\alpha_2)\in\mathcal{O}_{\mathcal{C},x}$ if and only if $\Phi_x([\alpha_1]_p)=[\alpha_2]_p$. For every $\alpha\in\mathcal{O}_{\mathcal{C}_1,x}$ we have $\Phi_x(\alpha)_{|C}=\alpha_{|C}$, and $\Phi_x(\pi_1)=\pi_2$.

The simplest case is $p=1$. In this case $\Phi:C\to C$ is the identity and $\mathcal{C}=\mathcal{A}$ (the *initial* gluing).

4.2.3. *Converse* - Recall that $\boldsymbol{\mathcal{A}}$ denotes the initial gluing of $\mathcal{C}_1,\mathcal{C}_2$ (cf. 4.1.5). Let $\Phi:\mathcal{C}_1^{(p-1)}\to\mathcal{C}_2^{(p-1)}$ be an isomorphism inducing the identity on C and such that $\Phi(\pi_1)=\pi_2$. We define a subsheaf of algebras $\mathcal{U}_\Phi$ of $\mathcal{O}_{\boldsymbol{\mathcal{A}}}$: $\mathcal{U}_\Phi=\mathcal{O}_{\boldsymbol{\mathcal{A}}}$ on $\boldsymbol{\mathcal{A}}\backslash C$, and for every point x of C

$$\mathcal{U}_{\Phi,x}\;=\;\{(\alpha_1,\alpha_2)\in\mathcal{O}_{\mathcal{C}_1,x}\times\mathcal{O}_{\mathcal{C}_2,x}\;;\;\Phi_x([\alpha_1]_p)=[\alpha_2]_p\}\;.$$

It is easy to see that $\mathcal{U}_\Phi$ is the structural sheaf of an algebraic variety $\mathcal{A}_\Phi$, that the inclusion $\mathcal{U}_\Phi\subset\mathcal{O}_{\boldsymbol{\mathcal{A}}}$ defines a dominant morphism $\boldsymbol{\mathcal{A}}\to\mathcal{A}_\Phi$ inducing an isomorphism between the underlying topological spaces (for the Zariski topology), and that the composed morphisms $\mathcal{C}_i\subset\boldsymbol{\mathcal{A}}\to\mathcal{A}_\Phi$, $i=1,2$, are immersions. Moreover, the morphism $\pi:\boldsymbol{\mathcal{A}}\to S$ factorizes through $\mathcal{A}_\Phi$:

$$\begin{tikzcd}\boldsymbol{\mathcal{A}} \arrow[r] \arrow[rr, bend right, "\pi"'] & \mathcal{A}_\Phi \arrow[r, "\pi_\Phi"] & S\end{tikzcd}$$

and $\pi_\Phi:\mathcal{A}_\Phi\to S$ is flat.

For $2\leq i\leq p$, let $\Phi^{(i)}:\mathcal{C}_1^{(i)}\to\mathcal{C}_2^{(i)}$ be the isomorphism induced by Φ.

4.2.4. Proposition: $\pi_\Phi^{-1}(P)$ *is a primitive double curve.*

Proof. Let x be a closed point of C. We first show that $\mathcal{I}_{C,x}^2 \subset (\pi)$. Let $u = (\pi_1\alpha, \pi_2\beta) \in \mathcal{I}_{C,x}$. Let $\beta' \in \mathcal{O}_{\mathcal{C}_2,x}$ be such that $\Phi_x([\alpha]_p) = [\beta']_p$. We have then $v = (\alpha, \beta') \in \mathcal{O}_{\mathcal{C},x}$. We have $u - \pi v = (0, \pi_2(\beta - \beta')) \in \mathcal{O}_{\mathcal{C},x}$. Therefore $[\pi_2(\beta - \beta')]_p = \Phi_x(0) = 0$. Hence $\pi_2(\beta - \beta') \in (\pi_2^p)$. We can then write

$$u \;=\; \pi v + (0, \pi_2^p\gamma).$$

Let $u' \in \mathcal{I}_{C,x}$, that can be written as $u' = \pi v' + (0, \pi_2^p\gamma')$. We have then

$$uu' \;=\; \pi.\big(\pi vv' + (0,\pi_2\gamma')v + (0,\pi_2\gamma)v' + (0,\pi_2^{2p-1}\gamma\gamma')\big) \;\in\; (\pi).$$

It remains to show that $\mathcal{I}_{C,x}/(\pi) \simeq \mathcal{O}_{C,x}$. We have

$$\begin{aligned}
\mathcal{I}_{C,x} &= \{(\pi_1\alpha,\pi_2\beta) \in \mathcal{O}_{\mathcal{C}_1,x}\times\mathcal{O}_{\mathcal{C}_2,x}; \Phi_x([\pi_1\alpha]_p) = [\pi_2\beta]_p\} \\
&= \{(\pi_1\alpha,\pi_2\beta) \in \mathcal{O}_{\mathcal{C}_1,x}\times\mathcal{O}_{\mathcal{C}_2,x}; \Phi_x^{(p-1)}([\alpha]_{p-1}) = [\beta]_{p-1}\}, \\
(\pi)_x &= \{(\pi_1\alpha,\pi_2\beta) \in \mathcal{O}_{\mathcal{C}_1,x}\times\mathcal{O}_{\mathcal{C}_2,x}; \Phi_x([\alpha]_p) = [\beta]_p\}.
\end{aligned}$$

So if $(\pi_1\alpha, \pi_2\beta) \in \mathcal{I}_{C,x}$, we have $w = \Phi_x([\alpha]_p) - [\beta]_p \in (\pi_2^{p-1})_x/(\pi_2^p)_x \simeq \mathcal{O}_{C,x}$. Hence we have a morphism of $\mathcal{O}_{\mathcal{C},x}$-modules

$$\begin{aligned}
\lambda : \mathcal{I}_{C,x} &\longrightarrow \mathcal{O}_{C,x} \\
(\pi_1\alpha,\pi_2\beta) &\longmapsto w
\end{aligned}$$

whose kernel is $(\pi)_x$. We have now only to show that λ is surjective, which follows from the fact that $\lambda(\pi_1^p, 0) = 1$. □

4.3. Spectrum of a fragmented deformation and ideals of sub-deformations

Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of C_n, $\mathcal{C}_1, \ldots, \mathcal{C}_n$ the irreducible components of $\mathcal{C}$. For $1 \le i \le n$, let $\pi_i = \pi_{|\mathcal{C}_i}$. As in 4.2, we denote also $t \circ \pi_i$ by π_i. Let $I = \{i, j\}$ be a subset of $\{1, \ldots, n\}$, with $i \neq j$. Then $\pi : \mathcal{C}_I \to S$ is a fragmented deformation of C_2. According to 4.2 there exists a unique integer $p > 0$ such that $\mathcal{I}_{C,\mathcal{C}_I}/(\pi)$ is generated by the image of $(\pi_i^p, 0)$ (and also by the image of $(0, \pi_j^p)$). Recall that p is the smallest integer q such that $\mathcal{I}_{C,\mathcal{C}_I}$ contains a non zero element of the form $(\pi_i^q\lambda, 0)$ (or $(0, \pi_j^q\mu)$), with $\lambda_{|C} \neq 0$ (resp. $\mu_{|C} \neq 0$). Let

$$p_{ij} \;=\; p_{ji} \;=\; p,$$

and $p_{ii} = 0$ for $1 \le i \le n$. The symmetric matrix $(p_{ij})_{1\le i,j\le n}$ is called the *spectrum* of $\mathcal{C}$.

4.3.1. *Generators of* $(\mathcal{I}_C^p + (\pi))/(\mathcal{I}_C^{p+1} + (\pi))$ - Let $i, j \in \{1, \ldots, n\}$ be such that $i \neq j$. Let $x \in C$. Since $\mathcal{C}_{\{i,j\}} \subset \mathcal{C}$ there exists an element $\mathbf{u}_{ij} = (u_m)_{1\le m\le n}$ of $\mathcal{O}_{\mathcal{C},x}$ such that $u_i = 0$ and $u_j = \pi_j^{p_{ij}}$. According to proposition 4.1.4, the image of $\mathbf{u}_{ij}$ generates $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$ at x.

According to proposition 4.2.1 and the fact that the image of $\mathbf{u}_{ij}$ generates $\mathcal{I}_{C,\mathcal{C}_{ij},x}/(\mathcal{I}_{C,\mathcal{C}_{ij},x}^2 + (\pi))$, for every integer m such that $m \neq i, j$ and that $1 \le m \le n$, u_m is of the form $u_m = \alpha_{ij}^{(m)}\pi_m^{p_{im}}$, with $\alpha_{ij}^{(m)} \in \mathcal{O}_{\mathcal{C}_m,x}$ invertible. Let $\alpha_{ij}^{(i)} = 0$ and $\alpha_{ij}^{(j)} = 1$.

4.3.2. Proposition: $\mathbf{1} - \alpha_{ij|C}^{(m)}$ *is a non zero constant, uniquely determined and independent of* x.

2 − *Let* $\mathbf{a}_{ij}^{(m)} = \alpha_{ij|C}^{(m)} \in \mathbb{C}$. *Then we have, for all integers* i, j, k, m, q *such that* $1 \leq i, j, k, m, q \leq n$, $i \neq j$, $i \neq k$

$$\mathbf{a}_{ik}^{(m)}\mathbf{a}_{ij}^{(q)} \ = \ \mathbf{a}_{ik}^{(q)}\mathbf{a}_{ij}^{(m)} .$$

In particular we have $\mathbf{a}_{ij}^{(m)} = \mathbf{a}_{ik}^{(m)}\mathbf{a}_{ij}^{(k)}$ *and* $\mathbf{a}_{ij}^{(m)}\mathbf{a}_{im}^{(j)} = 1$.

Proof. Let $\mathbf{u}'_{ij}$ have the same properties as $\mathbf{u}_{ij}$. Then $\mathbf{v} = \mathbf{u}'_{ij} - \mathbf{u}_{ij} \in \mathcal{I}_{C,x}^2 + (\pi)$. So the image of $\mathbf{v}$ in $\mathcal{O}_{\mathcal{C}_{im},x}$ belongs to $\mathcal{I}_{C,\mathcal{C}_{im},x}^2 + (\pi)$. It follows that the m-th component of $\mathbf{v}$ is a multiple of $\pi_m^{p_{im}+1}$. Hence $\alpha_{ij|C}^{(m)}$ is uniquely determined. It follows that when x varies the $\alpha_{ij|C}^{(m)}$ can be glued together and define a global section of $\mathcal{O}_C$, which must be a constant. This proves 1-.

Now we prove 2-. There exists $u \in \mathcal{O}_{\mathcal{C},x}$ such that the k-th component of u is $\alpha_{ij}^{(k)}$, and u is invertible. Then the image of $(v_m) = \frac{\mathbf{u}_{ij}}{u}$ generates $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$, and $v_k = 1$. Hence according to 1-, we have $v_{m|C} = \mathbf{a}_{ik}^{(m)}$, i.e.

$$\frac{\mathbf{a}_{ij}^{(m)}}{\mathbf{a}_{ij}^{(k)}} \ = \ \mathbf{a}_{ik}^{(m)} .$$

We have the same equality with q instead of m, whence 2- is easily deduced. □

Let p be an integer such that $1 \leq p < n$, and $(i_1, j_1), \ldots, (i_p, j_p)$ p pairs of distinct integers of $\{1, \ldots, n\}$. Then the image of $\prod_{m=1}^{p} \mathbf{u}_{i_m j_m}$ is a generator of $(\mathcal{I}_C^p + (\pi))/(\mathcal{I}_C^{p+1} + (\pi))$.

Let $I \subset \{1, \cdots, n\}$ be a nonempty subset, distinct from $\{1, \cdots, n\}$. Let $i \in \{1, \cdots, n\}\backslash I$. Let

$$\mathbf{u}_{I,i} \ = \ \prod_{j \in I} \mathbf{u}_{ji} \ .$$

Recall that $\mathcal{C}_I = \cup_{j \in I} \mathcal{C}_j \subset \mathcal{C}$.

4.3.3. Proposition: *The ideal sheaf of* $\mathcal{C}_I$ *is generated by* $\mathbf{u}_{I,i}$ *at* x.

Proof. According to proposition 4.1.6 there exists an embedding of a neighborhood of x in a smooth variety of dimension 3. In this variety each $\mathcal{C}_i$ is a smooth surface defined by a single equation. The ideal of the union of the $\mathcal{C}_i$, $i \in I$ is the product of these equations. □

4.3.4. Proposition: *Let* i, j, k *be distinct integers such that* $1 \leq i, j, k \leq n$. *Then if* $p_{ij} < p_{jk}$, *we have* $p_{ik} = p_{ij}$.

Proof. We can come down to the case $n = 3$ by considering $\mathcal{C}_{\{i,j,k\}}$. We can suppose that $p_{23} \leq p_{12} \leq p_{13}$, and we must show that $p_{23} = p_{12}$. We have

$$\mathbf{u}_{21} = (\pi_1^{p_{12}}, 0, \alpha_{21}^{(3)}\pi_3^{p_{23}}), \quad \mathbf{u}_{31} = (\pi_1^{p_{13}}, \alpha_{31}^{(2)}\pi_2^{p_{23}}, 0).$$

So

$$\mathbf{u}_{31} - \pi^{p_{13}-p_{12}}\mathbf{u}_{21} = \big(0, \alpha_{31}^{(2)}\pi_2^{p_{23}}, -\alpha_{21}^{(3)}\pi_3^{p_{23}+p_{13}-p_{12}}\big) \ \in \ \mathcal{O}_{\mathcal{C},x}.$$

Taking the image of this element in $\mathcal{O}_{\mathcal{C}_{12}x}$, we see that $p_{23} \geq p_{12}$, hence $p_{23} = p_{12}$. □

4.3.5. Proposition: **1** − *Let i, j be distinct integers such that $1 \leq i, j \leq n$. Then we have $\mathcal{I}_{C,x} = (\mathbf{u}_{ij}) + (\pi)$.*

2 − *Let $v = (v_m)_{1 \leq m \leq n} \in \mathcal{I}_{C,x}$ such that v_i is a multiple of π_i^p, with $p > 0$. Then we have $v \in (\mathbf{u}_{ij}) + (\pi^p)$.*

Proof. Let $N = 1 + \max_{1 \leq k \leq n}(q_i)$, where $q_i = \displaystyle\sum_{j=1}^{n} p_{ij}$. For every integer j such that $1 \leq j \leq n$ we have $(0, \ldots, 0, \pi_j^{q_j}, 0, \ldots, 0) \in \mathcal{O}_{\mathcal{C}}(\mathcal{C})$. Hence $\mathcal{I}_C^N \subset (\pi)$. We will show by induction on k that $\mathcal{I}_{C,x} \subset (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{C,x}^k$. Taking $k = N$ we obtain 1-.

For $k = 1$ it is obvious. Suppose that it is true for $k - 1 \geq 1$. It is enough to prove that $\mathcal{I}_{C,x}^{k-1} \subset (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{C,x}^k$. Let $w_1, \ldots, w_{k-1} \in \mathcal{I}_{C,x}$. Since the image of $\mathbf{u}_{ij}$ generates $\mathcal{I}_{C,x}/(\mathcal{I}_{C,x}^2 + (\pi))$, we can write w_p as

$$w_p = \lambda_p \mathbf{u}_{ij} + \pi\mu_p + \nu_p,$$

with $\lambda_p, \mu_p \in \mathcal{O}_{\mathcal{C},x}$ and $\nu_p \in \mathcal{I}_{C,x}^2$. So we have

$$w_1 \cdots w_{k-1} = \lambda \mathbf{u}_{ij} + \pi\mu + \nu,$$

with $\lambda, \mu \in \mathcal{O}_{\mathcal{C},x}$ and $\nu \in \mathcal{I}_{C,x}^{2k-2}$. Since $2k - 2 \geq k$, we have $w_1 \cdots w_{k-1} \in (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{C,x}^k$. This proves 1-.

We prove 2- by induction on p. The case $p = 1$ follows 1-. Suppose that it is true for $p - 1 \geq 1$. So we can write v as

$$v = \lambda \mathbf{u}_{ij} + \pi^{p-1}\mu,$$

with $\lambda, \mu \in \mathcal{O}_{\mathcal{C},x}$. We can write v_i as $v_i = \alpha\pi^p$. So we have $\alpha\pi_i^p = \pi_i^{p-1}\mu_i$, whence $\mu_i = \alpha\pi_i$. Hence $\mu \in \mathcal{I}_{Cx}$. According to 1- we can write μ as $\mu = \theta\mathbf{u}_{ij} + \pi\tau$, with $\theta, \tau \in \mathcal{O}_{\mathcal{C},x}$. So

$$v = (\lambda + \pi^{p-1}\theta)\mathbf{u}_{ij} + \pi^p\tau,$$

which proves the result for p. □

4.3.6. *The ideal sheaves $\mathcal{I}_{\mathcal{C}_I}$* – Recall that $I \subset \{1, \cdots, n\}$ is a nonempty subset, distinct from $\{1, \cdots, n\}$. For every subset J of $\{1, \cdots, n\}$, let $J^c = \{1, \cdots, n\} \backslash J$ and $\mathcal{O}_J = \mathcal{O}_{\mathcal{C}_J}$. It follows from proposition 4.3.3 that $\mathcal{I}_{\mathcal{C}_I}$ is a line bundle on $\mathcal{C}_{I^c}$.

From now on, we suppose that $S \subset \mathbb{C}$ and $P = 0$ (cf. proposition 3.2.6).

4.3.7. Theorem: *We have $\mathcal{I}_{\mathcal{C}_I} \simeq \mathcal{O}_{I^c}$.*

Proof. By induction on n. If $n = 2$ the result follows from proposition 4.2.1 and the fact that $S \subset \mathbb{C}$. Suppose that it is true for $n - 1 \geq 2$. We will prove that it is true for n by induction on the number of elements q of I^c. Suppose first that $q = 1$ and let i be the unique element of I^c. Then according to proposition 4.3.3, $\mathcal{I}_{\mathcal{C}_I}$ is generated by $(0, \ldots, 0, \pi_i^{q_i}, 0, \ldots, 0)$, so the result is true in this case. Suppose that it is true if $1 \leq q < k < n$, and that $q = k$. Let $K = \{1, \cdots, n-1\}$. We can assume that $I \subset K$.

According to proposition 4.3.3, we have, for every $x \in C$, $\mathcal{I}_{\mathcal{C}_I,x} \simeq \mathcal{O}_{I^c x}$. We have $\mathcal{I}_{\mathcal{C}_K} \subset \mathcal{I}_{\mathcal{C}_I}$, and $\mathcal{I}_{\mathcal{C}_K} \simeq \mathcal{O}_{\{n\}}$. We have

$$\mathcal{I}_{\mathcal{C}_I}/\mathcal{I}_{\mathcal{C}_K} = \mathcal{I}_{\mathcal{C}_I,\mathcal{C}_K}$$

(the ideal sheaf of $\mathcal{C}_I$ in $\mathcal{C}_K$). From the first induction hypothesis we have

$$\mathcal{I}_{\mathcal{C}_I,\mathcal{C}_K} \simeq \mathcal{O}_{(I\cup\{n\})^c}.$$

So we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\{n\}} \longrightarrow \mathcal{I}_{\mathcal{C}_I} \longrightarrow \mathcal{O}_{I^c\backslash\{n\}} \longrightarrow 0.$$

Now we will compute $\mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}})$. According to [6], 2.3, we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}) \longrightarrow \mathrm{Hom}(\mathrm{Tor}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}).$$

Since $\mathrm{Tor}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{I^c})$ is concentrated on $\mathcal{C}_{I^c\backslash\{n\}}$, we have

$$\mathrm{Hom}(\mathrm{Tor}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}) = \{0\}.$$

So we have

$$\mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}) = \mathrm{Ext}^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}).$$

Let $\mathcal{J}$ denote the ideal sheaf of $\mathcal{C}_{\{n\}}$ in $\mathcal{C}_{I^c}$. The ideal sheaf of $\mathcal{C}_{I^c\backslash\{n\}}$ is generated by $\mathbf{w} = (0, \dots, 0, \pi_n^m)$, with $m = \sum_{i\in I^c\backslash\{n\}} p_{in}$. So we have an exact sequence of sheaves on $\mathcal{C}_{I^c}$

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{I^c} \xrightarrow{\alpha} \mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c\backslash\{n\}} \longrightarrow 0\ ,$$

where α is the multiplication by $\mathbf{w}$. By the induction hypothesis there exists a surjective morphism $\mathcal{O}_{I^c} \to \mathcal{J}$, so we get a locally free resolution of $\mathcal{O}_{I^c\backslash\{n\}}$

$$\mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c} \xrightarrow{\alpha} \mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c\backslash\{n\}} \longrightarrow 0\ ,$$

that can be used to compute $\mathcal{E}xt^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}})$. It follows easily that

$$\mathcal{E}xt^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}) \simeq \mathcal{O}_{\{n\}}/(\pi_n^m)\ .$$

We have $\mathcal{H}om(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}) = 0$, hence

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}) &\simeq H^0(\mathcal{E}xt^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}})) \\ &\simeq H^0(\mathcal{O}_{\{n\}}/(\pi_n^m)) \\ &\simeq H^0(\mathcal{O}_S/(\pi_n^m)) \\ &\simeq \mathbb{C}[\pi_n]/(\pi_n^m).\end{aligned}$$

We will now describe the sheaves $\mathcal{E}$ such that there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_{\{n\}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{I^c\backslash\{n\}} \longrightarrow 0. \tag{2}$$

Let $\nu \in \mathbb{C}[\pi_n]/(\pi_n^m)$ be associated to this exact sequence, and $\overline{\nu} \in H^0(\mathcal{O}_S)$ over ν. Let

$$\begin{aligned}\tau : \mathcal{O}_{\{n\}} &\longrightarrow \mathcal{O}_{\{n\}} \oplus \mathcal{O}_{I^c} \\ u &\longmapsto (\overline{\nu}u, \mathbf{w}u)\end{aligned}$$

Then according to the preceding resolution of $\mathcal{O}_{I^c\backslash\{n\}}$ and the construction of extensions (cf. [5], 4.2), we have $\mathcal{E} \simeq \mathrm{coker}(\tau)$. It is easy to see that if $\nu = -1$ then $\mathcal{E} \simeq \mathcal{O}_{\mathcal{I}^c}$. If ν is invertible, then we have also $\mathcal{E} \simeq \mathcal{O}_{\mathcal{I}^c}$, because the corresponding extension can be obtained from the one corresponding to $\nu = -1$ by multiplying the left morphism of the exact sequence by ν.

A similar construction can be done for extensions of $\mathcal{O}_{I^c,x}$-modules (for every $x \in C$)

$$0 \longrightarrow \mathcal{O}_{\{n\},x} \longrightarrow V \longrightarrow \mathcal{O}_{I^c\backslash\{n\},x} \longrightarrow 0.$$

These extensions are classified by $\mathcal{O}_{\{n\},x}/(\pi_n^m)$, and $\mathcal{O}_{I^c,x}$ corresponds to -1.

Conversely we consider extensions

$$0 \longrightarrow \mathcal{O}_{\{n\},x} \xrightarrow{\lambda} \mathcal{O}_{I^c,x} \xrightarrow{\mu} \mathcal{O}_{I^c\backslash\{n\},x} \longrightarrow 0 \ .$$

Using the facts that $\mathrm{Hom}(\mathcal{O}_{\{n\},x}, \mathcal{O}_{I^c,x})$ is generated by the multiplication by $\mathbf{w}$ and $\mathrm{Hom}(\mathcal{O}_{I^c,x}, \mathcal{O}_{I^c\backslash\{n\},x})$ by the restriction morphism, it is easy to see that λ, μ are unique up to multiplication by an invertible element of $\mathcal{O}_{I^c,x}$. Hence the elements of $\mathrm{Ext}^1_{\mathcal{O}_{I^c,x}}(\mathcal{O}_{I^c\backslash\{n\},x}, \mathcal{O}_{\{n\},x})$ corresponding to the preceding extensions are exactly the invertible elements of $\mathcal{O}_{\{n\},x}/(\pi_n^m)$.

It follows that the extensions (2) where $\mathcal{E}$ is locally free correspond to invertible elements of $\mathbb{C}[\pi_n]/(\pi_n^m)$, and we have seen that in this case we have $\mathcal{E} \simeq \mathcal{O}_{I^c}$. Hence we have $\mathcal{I}_{\mathcal{C}_I} \simeq \mathcal{O}_{I^c}$ and theorem 4.3.7 is proved. □

4.3.8. Corollary: *The ideal sheaf of $\mathcal{C}_I$ is globally generated by an element $\mathbf{u}_I$ such that for every integer i such that $1 \leq i \leq n$ and $i \notin I$, the i-th coordinate of $\mathbf{u}_I$ belongs to $H^0(\mathcal{O}_S)$.*

4.4. Properties of the fragmented deformations

We use the notations of 4.3.

Let i be an integer such that $1 \leq i \leq n$ and $J_i = \{1, \ldots, n\}\backslash\{i\}$. We denote by $\mathcal{B}$ the image of $\mathcal{O}_\mathcal{C}$ in $\prod_{1\leq j\leq n} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$; it is a sheaf of $\mathbb{C}$-algebras on C. Let $\mathcal{B}_i$ be the image of $\mathcal{O}_{\mathcal{C}_{J_i}}$ in $\prod_{1\leq j\leq n, j\neq i} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$; it is also a sheaf of $\mathbb{C}$-algebras on C. For every point x of C and every $\alpha = (\alpha_m)_{1\leq m\leq n}$ in $\prod_{1\leq j\leq n} \mathcal{O}_{\mathcal{C}_j,x}$, we denote by $b_i(\alpha)$ its image in $\prod_{1\leq j\leq n, j\neq i} \mathcal{O}_{\mathcal{C}_j,x}$ (obtained by forgetting the i-th coordinate of α).

If p, k are positive integers, with $k \leq n$, $x \in C$ and $\alpha \in \mathcal{O}_{\mathcal{C}_k,x}$, let $[\alpha]_p$ denote the image of α in $\mathcal{O}_{\mathcal{C}_k,x}/\pi_k^p$.

4.4.1. Proposition: *There exists a morphism of sheaves of algebras on C*

$$\Phi_i : \mathcal{B}_i \longrightarrow \mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i})$$

such that for every point x of C and all $(\alpha_m)_{1\leq m\leq n, m\neq i} \in \mathcal{O}_{\mathcal{C}_{J_i},x}$, $\alpha_i \in \mathcal{O}_{\mathcal{C}_i,x}$, we have $\alpha = (\alpha_m)_{1\leq m\leq n} \in \mathcal{O}_{\mathcal{C},x}$ if and only if $\Phi_{i,x}(b_i(\alpha)) = [\alpha_i]_{q_i}$.

Proof. Let $(\alpha_m)_{1\leq m\leq n, m\neq i} \in \mathcal{O}_{\mathcal{C}_{J_i}x}$. Since $\mathcal{C}_{J_i} \subset \mathcal{C}$, there exists $\alpha_i \in \mathcal{O}_{\mathcal{C}_i,x}$ such that $(\alpha_m)_{1\leq m\leq n} \in \mathcal{O}_{\mathcal{C},x}$. If $\alpha'_i \in \mathcal{O}_{\mathcal{C}_i}x$ has the same property, we have $(0, \ldots, 0, \alpha_i - \alpha'_i, 0, \ldots, 0) \in \mathcal{I}_{J_i x}$. So according to proposition 4.3.3, we have $[\alpha_i]_{q_i} = [\alpha'_i]_{q_i}$. Hence we have a well defined morphism of algebras $\theta_x : \mathcal{O}_{\mathcal{C}_{J_i},x} \to \mathcal{O}_{\mathcal{C}_{J_i}}/(\pi_i^{q_i})$ sending $(\alpha_m)_{1\leq m\leq n, m\neq i}$ to $[\alpha_i]_{q_i}$. If $j \in J_i$, we have, according to proposition 4.3.3, $\theta_x(0, \ldots, 0, \pi_j^{q_j}, 0, \ldots, 0) = 0$. Hence θ_x induces a morphism of algebras $\mathcal{B}_{i,x} \to \mathcal{O}_{\mathcal{C}_i,x}/(\pi_i^{q_i})$. □

The morphism Φ_i has the following properties: for every point x of C

(i) For every $\alpha=(\alpha_m)_{1\leq m\leq n,m\neq i}\in\mathcal{B}_{i,x}$, we have $\Phi_{i,x}(\alpha)_{|C}=\alpha_{m|C}$ for $1\leq m\leq n$, $m\neq i$.
(ii) We have $\Phi_{i,x}((\pi_m)_{1\leq m\leq n,m\neq i})=\pi_i$.
(iii) Let $j,k\in\{1,\cdots,n\}$ be such that i,j,k are distinct. Let $\mathbf{v}$ be the image of $\mathbf{u}_{jk}$ in $\mathcal{B}_i$. Then there exists $\lambda\in\mathcal{O}^*_{\mathcal{C}_i,x}$ such that $\Phi_{i,x}(\mathbf{v})=\lambda\pi_i^{p_{ij}}$.
(iv) Let j be an integer such that $1\leq j\leq n$ and $j\neq i$. Let $\mathbf{v}$ be the image of $\mathbf{u}_{ij}$ in $\mathcal{B}_{i,x}$. Then we have $\ker(\Phi_{i,x})=(\mathbf{v})$.

4.4.2. *Converse* - Let $\mathcal{C}'$ be a gluing of $\mathcal{C}_1,\ldots,\mathcal{C}_{i-1},\mathcal{C}_{i+1},\ldots,\mathcal{C}_n$ along C, which is a fragmented deformation of a primitive multiple curve of multiplicity $n-1$. Let $(p_{jk})_{1\leq j,k\leq n,j,k\neq i}$ be the spectrum of $\mathcal{C}'$. Let p_{ij}, $1\leq j\leq n, j\neq i$ be positive integers, and $p_{ii}=0$. For $1\leq j\leq n$, let $q_j=\displaystyle\sum_{1\leq k\leq n}p_{kj}$.

Let $\mathcal{B}_i$ be the image of $\mathcal{O}_{\mathcal{C}'}$ in $\prod_{1\leq j\leq n,j\neq i}\mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$ and

$$\Phi_i:\mathcal{B}_i\longrightarrow\mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i})$$

a morphism of sheaves of algebras on C satisfying properties (i), (ii), (iii) above. Let $\mathcal{A}$ be the subsheaf of algebras of $\boldsymbol{\mathcal{A}}$ defined by: $\mathcal{A}=\boldsymbol{\mathcal{A}}$ on $\boldsymbol{\mathcal{A}}_{top}\backslash C$, and for every point x of C, and every $\alpha=(\alpha_m)_{1\leq m\leq n}\in\prod_{m=1}^n\mathcal{O}_{\mathcal{C}_m,x}$, $\alpha\in\mathcal{A}_x$ if and only if $b_i(\alpha)\in\mathcal{B}_{i,x}$ and $\Phi_{i,x}(b_i(\alpha))=[\alpha_i]_{q_i}$.

It is easy to see that $\mathcal{A}$ is the structural sheaf of a gluing of $C_1,\ldots,C_n$ along C, which is a fragmented deformation of a primitive multiple curve of multiplicity n, and that $\mathcal{C}'=\mathcal{A}_{\{1,\ldots,i-1,i+1,\ldots,n\}}$.

We give now some applications of the preceding construction.

4.4.3. Corollary: *Let N be an integer such that $N\geq\max_{1\leq i\leq n}(q_i)$. Let $x\in C$, $\beta\in\mathcal{O}_{\mathcal{C}_1,x}\times\cdots\mathcal{O}_{\mathcal{C}_n,x}$ and $u\in\mathcal{O}_{\mathcal{C},x}$ such that $u_{|C}\neq 0$. Suppose that $[\beta u]_N\in\mathcal{O}_{\mathcal{C},x}/(\pi^N)$. Then we have $[\beta]_N\in\mathcal{O}_{\mathcal{C},x}/(\pi^N)$.*

Proof. By induction on n. It is obvious if $n=1$. Suppose that the lemma is true for $n-1$. Let $I=\{1,\ldots,n-1\}$. So we have $[\beta_{|\mathcal{C}_1\times\cdots\mathcal{C}_{n-1}}]_N\in\mathcal{O}_{\mathcal{C}_I,x}/(\pi_1,\ldots,\pi_{n-1})^N$ by the induction hypothesis. Let γ (resp. v) be the image of β (resp. u) in $\mathcal{B}_n$. To show that $[\beta]_N\in\mathcal{O}_{\mathcal{C},x}/(\pi^N)$ it is enough to verify that

$$\Phi_n(\gamma)\ =\ [\beta_n]_{q_n}.$$

We have $\Phi_n(\gamma v)\ =\ [\beta_n u_n]_{q_n}$ because $[\beta u]_N\in\mathcal{O}_{\mathcal{C},x}/(\pi^N)$, and $\Phi_n(v)=[u_n]_{q_n}$ because $u\in\mathcal{O}_{\mathcal{C},x}$. So we have

$$\Phi_n(\gamma)[u_n]_{q_n}\ =\ \Phi_n(\gamma)\Phi_n(v)\ =\ \Phi_n(\gamma v)\ =\ [\beta_n u_n]_{q_n}\ =\ [\beta_n]_{q_n}[u_n]_{q_n}.$$

Since $u_{|C}\neq 0$, $[u_n]_{q_n}$ is not a zero divisor in $\mathcal{O}_{\mathcal{C}_n,x}/(\pi_n^{q_n})$, so we have $\Phi_n(\gamma)=[\beta_n]_{q_n}$. □

4.4.4. Corollary: *Let $\mathbf{q}=\max_{1\leq i\leq n}(q_i)$ and p the number of integers i such that $1\leq i\leq n$ and $q_i=\mathbf{q}$. Then we have $p\geq 2$.*

Proof. Suppose that $q_i = \mathbf{q}$. Then we have $\pi_i^{q_i-1} \neq 0$ in $\mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i})$. Since $\pi_i = \Phi_i((\pi_m)_{1\leq m\leq n, m\neq i})$, we have $(\pi_m^{q_i-1})_{1\leq m\leq n, m\neq i} \neq 0$ in $\mathcal{B}_i$. So we cannot have $q_m < q_i$ for all the $m \neq i$. □

Let i be an integer such that $1 \leq i \leq n$,

$$\mathcal{H} = \prod_{1\leq j\leq n} (\pi_j^{q_j-1})/(\pi_j^{q_j}) \simeq \mathcal{O}_C^n \qquad (\text{resp.} \quad \mathcal{H}_i = \prod_{1\leq j\leq n, j\neq i} (\pi_j^{q_j-1})/(\pi_j^{q_j}) \simeq \mathcal{O}_C^{n-1}\ ).$$

It is an ideal sheaf of $\prod_{1\leq j\leq n} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$ (resp. $\prod_{1\leq j\leq n, j\neq i} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$). Let $\mathcal{J} = \mathcal{H} \cap \mathcal{B}$ (resp. $\mathcal{J}_i = \mathcal{H}_i \cap \mathcal{B}_i$), which is an ideal sheaf of $\mathcal{B}$ (resp. $\mathcal{B}_i$).

4.4.5. Proposition: *There exists a unique* $\lambda(\mathcal{C}) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}_n(\mathbb{C})$ *such that for every* $\mathbf{u} = (u_j)_{1\leq j\leq n} \in \mathcal{H}$, *we have* $\mathbf{u} \in \mathcal{J}$ *if and only if* $\lambda_1 u_1 + \cdots \lambda_n u_n = 0$. *The* λ_i *are all non zero.*

Proof. We have $(\pi_m)_{1\leq m\leq n, m\neq i}.\mathcal{J}_i = 0$. Hence $\pi_i \Phi_i(\mathcal{J}_i) = 0$ and $\Phi_i(\mathcal{J}_i) \subset (\pi_i^{q_i-1})/(\pi_j^{q_i})$. The restriction of Φ_i, $\mathcal{J}_i \to (\pi_i^{q_i-1})/(\pi_j^{q_i})$ is a morphism $(n-1)\mathcal{O}_C \to \mathcal{O}_C$ of vector bundles on C. The existence of $(\lambda_1, \ldots, \lambda_n)$ follows from that.

If $\lambda_i = 0$, we have $(0, \ldots, 0, \pi_i^{q_i-1}, 0, \ldots, 0) \in \mathcal{O}_\mathcal{C}(\mathcal{C})$. This is impossible because according to proposition 4.3.3, $(0, \ldots, 0, \pi_i^{q_i}, 0, \ldots, 0)$ generates the ideal sheaf of $\mathcal{C}_{J_i}$ in $\mathcal{C}$. □

For all distinct integers i, j such that $1 \leq i, j \leq n$, let $I_{ij} = \{1, \ldots, n\}\backslash\{i, j\}$. Then according to proposition 4.3.3, $\mathbf{u}_{I_{ij}i}$ generates the ideal sheaf of $\mathcal{C}_{I_{ij}}$. We have $\mathbf{u}_{I_{ij}i} = (b_k)_{1\leq k\leq n}$, with $b_k = 0$ if $k \neq i, j$, $b_i = \pi_i^{q_i-p_{ij}}$ and

$$b_j \ = \ \Big(\prod_{1\leq m\leq n, m\neq i,j} \alpha_{mi}^{(j)} \ \Big).\pi_j^{q_j-p_{ij}}.$$

So we have $\pi^{p_{ij}-1}\mathbf{u}_{I_{ij}i} \in \mathcal{J}_i$, which gives the equation

$$\frac{\lambda_i}{\lambda_j} \ = \ - \prod_{1\leq m\leq n, m\neq i,j} \mathbf{a}_{mi}^{(j)}. \tag{3}$$

4.4.6. Proposition: *For all distinct integers* i, j, k *such that* $1 \leq i, j, k \leq n$, *we have*

$$\mathbf{a}_{ki}^{(j)} \ = \ -\mathbf{a}_{ik}^{(j)}\mathbf{a}_{ji}^{(k)} \ .$$

Proof. We need only to treat the case $n = 3$, and we get the preceding formula by writing that $\frac{\lambda_1}{\lambda_3} = \frac{\lambda_1}{\lambda_2}.\frac{\lambda_2}{\lambda_3}$, and by using (3). □

4.4.7. Proposition: *Let* $(\alpha_1\pi_1^{m_1}, \ldots, \alpha_n\pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C},x}$, *with* $\alpha_1, \ldots, \alpha_n$ *invertible. Let* $M = m_1 + \cdots + m_n$. *then*

$$\Big(\frac{1}{\alpha_1}\pi_1^{M-m_1}, \ldots, \frac{1}{\alpha_n}\pi_n^{M-m_n}\Big) \ \in \ \mathcal{O}_{\mathcal{C},x} \ .$$

Proof. By induction on n. It is obvious for $n=1$. Suppose that it is true for $n-1\geq 1$. Let $I=\{1,\ldots,n-1\}$. Then $(\alpha_1\pi_1^{m_1},\ldots,\alpha_{n-1}\pi_{n-1}^{m_{n-1}})\in\mathcal{O}_{\mathcal{C}_{I,x}}$. Hence, by the induction hypothesis, we have

$$\big(\frac{1}{\alpha_1}\pi_1^{M-m_1-m_n},\ldots,\frac{1}{\alpha_{n-1}}\pi_{n-1}^{M-m_{n-1}-m_n}\big)\ \in\ \mathcal{O}_{\mathcal{C}_{I,x}}\ .$$

So there exists $\gamma\in\mathcal{O}_{\mathcal{C}_{n,x}}$ such that

$$u\ =\ \big(\frac{1}{\alpha_1}\pi_1^{M-m_1-m_n},\ldots,\frac{1}{\alpha_{n-1}}\pi_{n-1}^{M-m_{n-1}-m_n},\gamma\big)\ \in\ \mathcal{O}_{\mathcal{C},x}\ .$$

Multiplying by $(\alpha_1\pi_1^{m_1},\ldots,\alpha_n\pi_n^{m_n})$ we see that $(\pi_1^{M-m_n},\ldots,\pi_{n-1}^{M-m_n},\gamma\alpha_n\pi_n^{m_n})\in\mathcal{O}_{\mathcal{C},x}$. Subtracting π^{M-m_n}, we find that $(0,\ldots,0,\gamma\alpha_n\pi_n^{m_n}-\pi_n^{M-m_n})\in\mathcal{O}_{\mathcal{C},x}$. There exists $\alpha\in\mathcal{O}_{\mathcal{C},x}$ such that the n-th coordinate of α is α_n, and α is invertible. It follows that $v=(0,\ldots,0,\gamma\pi_n^{m_n}-\frac{1}{\alpha_n}\pi_n^{M-m_n})\in\mathcal{O}_{\mathcal{C},x}$. Now we have

$$\pi^{m_n}u-v\ =\ \big(\frac{1}{\alpha_1}\pi_1^{M-m_1},\ldots,\frac{1}{\alpha_n}\pi_n^{M-m_n}\big)\ \in\ \mathcal{O}_{\mathcal{C},x}\ .$$

□

4.4.8. Corollary: *Let $V\subset U$ be open subsets of $\mathcal{C}$, and suppose that $U\cap C\neq\emptyset$. Let $\alpha\in\mathcal{O}_{\mathcal{C}}(V)$ and $\beta\in\mathcal{O}_{\boldsymbol{\mathcal{A}}}(U)$ such that $\beta_{|V}=\alpha$. Then $\beta\in\mathcal{O}_{\mathcal{C}}(U)$.*

(Recall that $\boldsymbol{\mathcal{A}}$ is the *initial gluing* of $\mathcal{C}_1,\ldots,\mathcal{C}_n$ (cf. 4.1.5)).

Proof. This can be proved easily by induction on n, using proposition 4.4.1. □

4.5. Construction of fragmented deformations

Consider a fragmented deformation

$$\pi=\pi^{[n-1]}=(\pi_1,\ldots,\pi_{n-1}):\mathcal{C}^{[n-1]}\longrightarrow S$$

of C_{n-1}, with $n-1$ irreducible components $\mathcal{C}_1,\ldots,\mathcal{C}_{n-1}$. Let $(p_{ij}^{[n-1]})_{1\leq i,j<n}$ be its spectrum. For $1\leq i<n$, let $q_i^{[n-1]}=\displaystyle\sum_{1\leq j<n}p_{ij}^{[n-1]}$. We denote by $\mathcal{I}_C^{[n-1]}$ the ideal sheaf of C in $\mathcal{C}^{[n-1]}$. Let $\lambda(\mathcal{C}^{[n-1]})=(\lambda_1,\ldots,\lambda_{n-1})$.

Let $p_{1n},\ldots,p_{n-1,n}$ be positive integers, $q_i=q_i^{[n-1]}+p_{in}$ for $1\leq i<n$, and $q_n=p_{1n}+\cdots+p_{n-1,n}$. Let $\mathbf{u}\in H^0(\mathcal{I}_C^{[n-1]})$ whose image generates $\mathcal{I}_C^{[n-1]}/((\mathcal{I}_C^{[n-1]})^2+(\pi))$, of the form

$$\mathbf{u}\ =\ (\beta_1\pi_1^{p_{1n}},\ldots,\beta_{n-1}\pi_{n-1}^{p_{n-1,n}}),$$

with $\beta_i\in H^0(\mathcal{O}_S)$ invertible for $1\leq i<n$.

Let $\mathcal{B}^{[n-1]}$ be the image of $\mathcal{O}_{\mathcal{C}^{[n-1]}}$ in $\mathcal{O}_{\mathcal{C}_1}/(\pi_1^{q_1})\times\cdots\times\mathcal{O}_{\mathcal{C}_{n-1}}/(\pi_{n-1}^{q_{n-1}})$. We will also denote by $\mathbf{u}$ the image of $\mathbf{u}$ in $\mathcal{B}^{[n-1]}$. Let $\mathcal{Q}=\mathcal{B}^{[n-1]}/(\mathbf{u})$, $\rho:\mathcal{B}^{[n-1]}\to\mathcal{Q}$ the projection and $\pi_n=\rho(\pi)$.

4.5.1. Proposition: *We have $\pi_n^{q_n}=0$.*

Proof. According to proposition 4.4.7 we have for every $x\in C$

$$v \;=\; \big(\frac{1}{\beta_1}\pi_1^{q_n-p_{1n}},\dots,\frac{1}{\beta_{n-1}}\pi_n^{q_n-p_{n-1,n}}\big) \;\in\; \mathcal{O}_{\mathcal{C}^{[n-1]}x} \ .$$

Hence $\pi^{q_n}=v\mathbf{u}\in(\mathbf{u})$ in $\mathcal{O}_{\mathcal{C}^{[n-1]}x}$, and $\pi_n^{q_n}=0$. □

4.5.2. Proposition: 1 − *We have* $\pi_n^{q_n-1}=0$ *if and only if*

$$\frac{\lambda_1}{\beta_{1|C}}+\cdots+\frac{\lambda_{n-1}}{\beta_{n-1|C}} \;=\; 0 \ .$$

We suppose now that $\frac{\lambda_1}{\beta_{1|C}}+\cdots+\frac{\lambda_{n-1}}{\beta_{n-1|C}}\neq 0$*. Let* $x\in C$*. Then*

2 − *For every* $\epsilon\in\mathcal{B}_x^{[n-1]}$ *such that* $\epsilon_{|C}\neq 0$*, we have* $\pi^{q_n-1}\epsilon\notin(\mathbf{u})$*.*

3 − *For every* $\eta\in\mathcal{B}_x^{[n-1]}/(\mathbf{u})$*, and every integer* k *such that* $1\le k<q_n$*, we have* $\pi_n^k\eta=0$ *if and only if* η *is a multiple of* $\pi_n^{q_n-k}$*.*

4 − $\mathcal{B}_x^{[n-1]}/(\mathbf{u})$ *is a flat* $\mathbb{C}[\pi_n]/(\pi_n^{q_n})$*-module.*

Proof. We have $\pi_n^{q_n-1}=0$ if and only if $(\pi_1^{q_n-1},\dots,\pi_{n-1}^{q_n-1})\in(\mathbf{u})$ in $\mathcal{B}_x^{[n-1]}$. We have, in $\mathcal{O}_{\mathcal{C}_{1x}}\times\cdots\times\mathcal{O}_{\mathcal{C}_{n-1,x}}$,

$$(\pi_1^{q_n-1},\dots,\pi_{n-1}^{q_n-1}) \;=\; (\beta_1\pi_1^{p_{1n}},\dots,\beta_{n-1,n}\pi_{n-1}^{p_{n-1,n}}).(\frac{1}{\beta_1}\pi_1^{q_1^{[n-1]}-1},\dots,\frac{1}{\beta_p}\pi_1^{q_{n-1}^{[n-1]}-1}),$$

and $\pi_n^{q_n-1}=0$ if and only if there exist $\eta\in\mathcal{O}_{\mathcal{C}^{[n-1]},x}$, $a_i\in\mathcal{O}_{\mathcal{C}_i,x}$, $1\le i<n$, such that

$$(\pi_1^{q_n-1},\dots,\pi_{n-1}^{q_n-1}) \;=\; \eta\mathbf{u}+(a_1\pi_1^{q_1},\dots,a_{n-1}\pi_{n-1}^{q_{n-1}}).$$

This equality is equivalent to

$$(\frac{1}{\beta_1}\pi_1^{q_1^{[n-1]}-1},\dots,\frac{1}{\beta_{n-1}}\pi_1^{q_{n-1}^{[n-1]}-1})-\eta \;=\; (\frac{a_1}{\beta_1}\pi_1^{q_1^{[n-1]}},\dots,\frac{a_{n-1}}{\beta_{n-1}}\pi_{n-1}^{q_{n-1}^{[n-1]}}).$$

Since for $1\le i<n$, we have $(0,\dots,0,\pi_i^{q_i^{[n-1]}},0,\dots0)\in\mathcal{O}_{\mathcal{C}^{[n-1]},x}$, we have $\pi_n^{q_n-1}=0$ if and only if

$$(\frac{1}{\beta_1}\pi_1^{q_1^{[n-1]}-1},\dots,\frac{1}{\beta_{n-1}}\pi_1^{q_{n-1}^{[n-1]}-1}) \;\in\; \mathcal{O}_{\mathcal{C}^{[n-1]},x}.$$

So the result of 1- follows from the definition of $\lambda(\mathcal{C}^{[n-1]})$ (cf. prop. 4.4.5), 2- is an easy consequence.

Now we prove 3-, by induction on k. Suppose that it is true for $k=1$, and that $\pi_n^k\eta=0$, with $2\le k<q_n$. We have $\pi_n^{k-1}.\pi_n\eta=0$, so according to the induction hypothesis, $\pi_n\eta$ is a multiple of $\pi_n^{q_n-k+1}$: $\pi_n\eta=\pi_n^{q_n-k+1}\lambda$. So $\pi_n(\eta-\pi_n^{q_n-k}\lambda)=0$. Since -3 is true for $k=1$, we can write $\eta-\pi_n^{q_n-k}\lambda=\pi_n^{q_n-1}\epsilon$, i.e. $\eta=\pi_n^{q_n-k}(\lambda+\pi_n^{k-1}\epsilon)$, and 3- is true for k.

Il remains to prove 3- for $k=1$. Suppose that $\pi_n\eta=0$ (with $\eta\neq 0$). We can write η as $\eta=\pi_n^m\theta$, where θ is not a multiple of π_n, and $0\le m<q_n$. Let $\overline{\theta}\in\mathcal{B}_x^{[n-1]}$ be over θ. Since $\mathcal{I}_C^{[n-1]}=(\mathbf{u})+(\pi)$ according to proposition 4.3.5, the condition "θ is not a multiple of π_n" is equivalent to $\overline{\theta}\notin\mathcal{I}_{C,x}^{[n-1]}$. We have $\pi^{m+1}\overline{\theta}\in(\mathbf{u})$, so according to 2-, we have $m+1\ge q_n$, which proves 3- for $k=1$. The last assertion is an easy consequence of 3-. □

4.5.3. Example: Let N be an integer, $s \in H^0(\mathcal{O}_S)$ invertible, and k, l integers such that $1 \leq k, l < n$, $k \neq l$. Suppose that for every integer i such that $1 \leq i < n$ and $i \neq k$ we have $N > p_{ik}^{[n-1]}$ and $N \geq q_i^{[n-1]} - q_k^{[n-1]} + p_{ik}^{[n-1]}$. We take $\mathbf{u} = \mathbf{u}_{kl} - s\pi^N$. We have then $\beta_i = \alpha_{kl}^{(i)}$ if $i \neq k$, and $\beta_k = -s$. The condition $\frac{\lambda_1}{\beta_{1|C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$ is fulfilled if and only if

$$\sum_{1 \leq i < n, i \neq k} \frac{\lambda_i}{\mathbf{a}_{kl}^{(i)}} - \frac{\lambda_k}{s_{|C}} \neq 0.$$

4.5.4. *Construction of fragmented deformations* – Suppose that $\frac{\lambda_1}{\beta_{1|C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$. From proposition 4.5.2, 4-, it is easy to prove that

- There exists a flat morphism of algebraic varieties $\tau : Y \to \operatorname{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$ with a canonical isomorphism of sheaves of $\mathbb{C}[\pi_n]/(\pi_n^{q_n})$-algebras $\mathcal{O}_Y \simeq \mathcal{Q}$, such that $\tau^{-1}(*) = C$ (where $*$ is the closed point of $\operatorname{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$).
- There exist a family of smooth curves $\mathcal{C}_n$ and a flat morphism $\pi_n : \mathcal{C}_n \to S$ extending τ (recall that S is a germ). Hence Y is the inverse image of the subscheme of $\mathcal{C}_n$ corresponding to the ideal sheaf $(\pi_n^{q_n})$. The existence of $\mathcal{C}_n$ can be proved using Hilbert schemes of curves in projective spaces. Of course $\mathcal{C}_n$ need not be unique.

We obtain a gluing $\mathcal{C}$ of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ by defining the sheaves of algebras $\mathcal{O}_{\mathcal{C}}$ (on the Zariski topological space corresponding to the initial gluing $\boldsymbol{\mathcal{A}}$) as in 4.4.2, using for Φ_n the quotient morphism $\mathcal{B}^{[n-1]} \to \mathcal{Q}$. It is easy to see that $\pi^{-1}(P)$ is a primitive multiple curve C_n of multiplicity n extending C_{n-1}, hence $\mathcal{C}$ is a fragmented deformation of C_n.

4.5.5. Remark: 1 – The multiple curve C_n depends on the choice of the family $\mathcal{C}_n$ extending the family Y parametrized by $\operatorname{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$.

2 – The multiple curve C_{n-1} is completely defined by $\mathcal{B}^{[n-1]}$, because $(\pi_1^{q_1}) \times \cdots (\pi_{n-1}^{q_{n-1}}) \subset (\pi)$. But it is not enough to know $\mathcal{B}^{[n-1]}$ and $\mathbf{u}$ to define C_n. In fact we need $\mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i+1})$, $1 \leq i \leq n$.

4.6. Basic elements

We use the notations of 4.3 and 4.4.

Let $\mathbf{m} = (m_1, \ldots, m_n)$ be an n-tuple of positive integers, and

$$\mathbf{\Pi^m} = (\pi_1^{m_1}) \times \cdots \times (\pi_n^{m_n}).$$

4.6.1. Definition: *Let $x \in C$. An element u of $\mathcal{O}_{\mathcal{C},x}$ is called* basic at order $\mathbf{m}$ *if there exist polynomials $P_1, \ldots, P_n \in \mathbb{C}[X]$ such that*

$$u \equiv (P_1(\pi_1), \ldots, P_n(\pi_n)) \qquad (\text{mod.} \quad \mathbf{\Pi^m})\ .$$

If $u = (P_1(\pi_1), \ldots, P_n(\pi_n))$, we say that u is basic.

Let $\mathbf{q} = (q_1, \ldots, q_n)$. Then according to corollary 4.4.8, if u is basic at order $\mathbf{q}$, then for every $y \in C$, we have $(P_1(\pi_1), \ldots, P_n(\pi_n)) \in \mathcal{O}_{\mathcal{C},y}$. So $(P_1(\pi_1), \ldots, P_n(\pi_n))$ is defined on a neighborhood of C.

4.6.2. Lemma: *Let $u, v, w \in \mathcal{O}_{\mathcal{C},x}$ such that $w = uv$ and $w \neq 0$. Suppose that u and w are basic at every order. Then v is basic at every order.*

Proof. Let N be a positive integer such that $N \gg 0$ and $\mathbf{N} = (N, \ldots, N)$. Suppose that $w \equiv (Q_1(\pi_1), \ldots, Q_n(\pi_n)) \pmod{(\pi^N)}$, where $Q_1, \ldots, Q_n \in \mathbb{C}[X]$. Let $\mathbf{m} = (m_1, \ldots, m_n)$ be an n-tuple of positive integers, and $v = (v_i)_{1 \leq i \leq n}$. Suppose that

$$u \ \equiv \ (P_1(\pi_1), \ldots, P_n(\pi_n)) \qquad (\text{mod.} \quad \mathbf{\Pi^N})$$

Then we have

$$Q_i(\pi_i) \ \equiv \ P_i(\pi_i).v_i \qquad (\text{mod.} \quad (\pi_i^N))$$

for $1 \leq i \leq n$. We can write $P_i(X)$ as $P_i(X) = X^{n_i} R_i(X)$, where $R_i(X) \in \mathbb{C}[X]$ is such that $R_i(0) \neq 0$. Then $Q_i(X)$ is also divisible by X^{n_i}: $Q_i(X) = X^{n_i} S_i(X)$, and we have in $\mathcal{O}_{\boldsymbol{\mathcal{A}}x}$:

$$S_i(\pi_i) \ \equiv \ R_i(\pi_i).v_i \qquad (\text{mod.} \quad (\pi_i^{N'}))$$

for some integer $N' \gg 0$. We can write $R_i(X) = a_i.(1 - X.T_i(X))$, with $a_i \in \mathbb{C}^*$, $T_i \in \mathbb{C}(X)$. We have then

$$v_i \ \equiv \ \frac{S_i(\pi_i)}{a_i} \sum_{p=1}^{m_i - 1} \big(\pi_i T_i(\pi_i)\big)^p \qquad (\text{mod.} \quad \mathbf{\Pi^m}).$$

□

For $1 \leq i \leq n$, let $\mathbf{u}_{(i)} = ((u_{(i)j})_{1 \leq j \leq n}$ be a generator of the ideal sheaf $\mathcal{I}_{\mathcal{C}_i}$ of $\mathcal{C}_i$ in $\mathcal{C}$, such that for $1 \leq j \leq n$, $u_{(i)j} \in \mathbb{C}[\pi_j]$ (cf. corollary 4.3.8).

4.6.3. Proposition: *Let $v \in \mathcal{O}_{\mathcal{C},x}$. then v is basic at every order if and only if for every n-tuple $\mathbf{m}$ of positive integers, there exist an integer $q > 0$ and $P_1, \ldots, P_q \in \mathbb{C}[X]$ such that*

$$v \ \equiv \sum_{1 \leq j \leq q} P_j(\pi).\mathbf{u}_{(i)}^j \qquad (\text{mod.} \quad \mathbf{\Pi^m}).$$

Proof. We use the notations of the proof of lemma 4.6.2. Suppose that $v = (v_j)_{1 \leq j \leq n}$ is basic at every order. Let N be a positive integer and $\mathbf{N} = (N, \ldots, N)$. We will prove by induction on $q \geq 0$ that we can write v as

$$v \ \equiv \sum_{0 \leq j \leq q} P_j(\pi).\mathbf{u}_{(i)}^j + \gamma_q \mathbf{u}_{(i)}^{q+1} \qquad (\text{mod.} \quad \mathbf{\Pi^N}) \tag{4}$$

with $P_0, \ldots, P_q \in \mathbb{C}[X]$, and $\gamma_q \in \mathcal{O}_{\mathcal{C},x}$. This proves proposition 4.6.3 if q and N are big enough. For $q = 0$, we have $v_i \equiv P(\pi_i)(\text{mod } \pi_i^N)$, for some $P \in \mathbb{C}[X]$, and we can take $P_0 = P$. Suppose that the result is true for q and that we have (4). Since $v - \sum_{1 \leq j \leq q} P_j(\pi).\mathbf{u}_{(i)}^j$ is basic at any order, using the same method as in the proof of lemma 4.6.2, we see that γ_q is basic at order $\mathbf{N}'$, where $\mathbf{N}' = (N', \ldots, N')$, for some integer $N' \gg 0$. As in the case $q = 0$ we have

$$\gamma_q \ \equiv \ P_{q+1}(\pi) + \mathbf{u}_{(i)}.\gamma_{q+1} \qquad (\text{mod} \quad \mathbf{\Pi^{N'}}),$$

with $P_{q+1} \in \mathbb{C}[X]$. Hence

$$v \ \equiv \sum_{0 \leq j \leq q+1} P_j(\pi).\mathbf{u}_{(i)}^j + \gamma_{q+1} \mathbf{u}_{(i)}^{q+2} \qquad (\text{mod.} \quad \mathbf{\Pi^N})$$

□

4.6.4. Proposition: *Let* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C},x}$ *be such that there exists* $P_1, \ldots, P_{n-1} \in \mathbb{C}[X]$ *such that, for* $1 \leq i \leq n-1$, *we have* $\alpha_i \equiv P_i(\pi_i)$ *(mod.* $(\pi_i^{q_i})$*). Then there exists* $P_n \in \mathbb{C}[X]$ *such that* $\alpha_n \equiv P_n(\pi_n)$ *(mod.* (π^{q_n})*), i.e.* α *is a basic element of order* $\mathbf{q}$.

Proof. By induction on n. The case $n = 2$ is an easy consequence of proposition 4.2.2. Suppose that $n \geq 3$ and that the result is true for $n-1$.

By subtracting multiples of $(0, \ldots, 0, \pi_i^{q_i}, 0, \ldots, 0)$ we may assume that for $1 \leq i \leq n-1$, $\alpha_i \in \mathbb{C}[\pi_i]$. By subtracting a regular function on a neighborhood of C in $\mathcal{C}$, and a multiple of $(\pi_1^{q_1}, 0, \ldots, 0)$ we may also assume that $\alpha_1 = 0$. The ideal sheaf of $\mathcal{C}_1$ is generated by $\mathbf{u}_{(1)}$. We can then write $\alpha = \beta\mathbf{u}_{(1)}$, with $\beta = (\beta_i)_{1 \leq i \leq n} \in \mathcal{O}_{\mathcal{C},x}$. We have

$$(\alpha_2, \ldots, \alpha_{n-1}) = (\beta_2, \ldots, \beta_{n-1}).(u_{(1)2}, \ldots, u_{(1)n-1}) \ ,$$

hence by lemma 4.6.2, $(\beta_2, \ldots, \beta_{n-1})$ is a basic element at any order. By the induction hypothesis, there exists $Q \in \mathbb{C}[X]$ such that $\beta_n \equiv Q(\pi_n)$ (mod. $(\pi_n^{q_n - p_{1n}})$). Since $u_{(1)n}$ is a multiple of $\pi_n^{p_{1n}}$ (from the definition of p_{1n}), it follows that $\alpha_n \equiv u_{(i)n}Q(\pi_n)$ (mod. (π^{q_n})). □

4.7. Simple primitive curves and fragmented deformations

Let C_n be a primitive multiple curve of multiplicity n and associated smooth curve C. Let $\mathcal{I}_C$ be the ideal sheaf of C in C_n. It is obvious from proposition 4.3.5, 1-, that if there exists a fragmented deformation of C_n, then we have $\mathcal{I}_{C,C_n} \simeq \mathcal{O}_{C_{n-1}}$, i.e. C_n is *simple* (cf. 2.4). Conversely we have

4.7.1. Theorem: *Let* C_n *be a simple primitive multiple curve of multiplicity* n. *Then there exists a fragmented deformation of* C_n.

Proof. According to theorem 2.4.1, there exists a flat family of smooth projective curves $\tau : \mathcal{C} \to \mathbb{C}$ such that $\tau^{-1}(0) \simeq C$ and that C_n is isomorphic to the n-th infinitesimal neighborhood of C in $\mathcal{C}$. Let $\rho_n : \mathbb{C} \to \mathbb{C}$ be the map defined by $\rho_n(z) = z^n$, and $\theta = \rho_n \circ \tau : \mathcal{C} \to \mathbb{C}$. It is a flat morphism, $\theta^{-1}(0) = C_n$, and for every $z \neq 0$ in the image of τ, $\theta^{-1}(z)$ is a disjoint union of n smooth irreducible curves. We can then apply the process of proposition 3.1.3 to obtain the desired fragmented deformation: it is $\mathcal{C} \times_{\mathbb{C}} \mathbb{C}$

$$\begin{tikzcd} \mathcal{C} \times_{\mathbb{C}} \mathbb{C} \arrow[r, "\pi"] \arrow[d] & \mathbb{C} \arrow[d, "\rho_n"] \\ \mathcal{C} \arrow[r, "\theta"] & \mathbb{C} \end{tikzcd}$$

□

4.7.2. Remark: let (p_{ij}) be the spectrum of the fragmented deformation constructed in the proof of theorem 4.7.1. Then it is easy to see that $p_{ij} = 1$ for $1 \leq i, j \leq n$, $i \neq j$. If $x \in C$, then

$(\mathcal{C}\times_{\mathbb{C}}\mathbb{C})_x=\mathcal{O}_{\mathcal{C},x}\otimes_{\mathcal{O}_{\mathbb{C},x}}\mathcal{O}_{\mathbb{C},x}$, and if $t=I_{\mathbb{C}}\in\mathcal{O}_{\mathbb{C},x}$, we have for $1\leq k\leq n$

$$(\pi_1,\ldots,\pi_{k-1},0,\pi_{k+1},\ldots,\pi_n) \ = \ \frac{1}{n-1}(1\otimes t-e^{\frac{2ki\pi}{n}}(t\otimes 1)) \ .$$

5. Stars of a curve

5.1. Definitions

Let S be a smooth irreducible curve, and $P\in S$ (we can also take for (S,P) the germ of a smooth curve). Let n be a positive integer.

5.1.1. Definition: *An* n-star *(or more simply, a* star*) of (S,P) is an algebraic variety $\boldsymbol{\mathcal{S}}$ such that*

(i) *$\boldsymbol{\mathcal{S}}$ is the union of n irreducible components $S_1,\ldots,S_n$, with fixed isomorphisms $S_i\simeq S$, $1\leq i\leq n$.*
(ii) *For $1\leq i<j\leq n$, $S_i\cap S_j$ has only one closed point, namely P.*
(iii) *There exists a morphism $\pi:\boldsymbol{\mathcal{S}}\to S$, such that for $1\leq i\leq n$, the restriction $\pi_{|S_i}:S_i\to S$ is the isomorphism $S_i\simeq S$ of (i).*

All the n-stars of (S,P) have the same underlying Zariski topological space $S(n)$ and set of closed points. The latter is $(\bigcup_{1\leq i\leq n}\widehat{S}_i)/\sim$, where $\widehat{S}_i$ is the set of closed points of S_i, and the equivalence relation $\sim$ is defined by: for $x\in\widehat{S}_i$ and $y\in\widehat{S}_j$, $x\sim y$ if and only if $i=j$ and $x=y$, or $x=P\in\widehat{S}_i$ and $y=P\in\widehat{S}_j$. An open subset of $\boldsymbol{\mathcal{S}}$ is defined by open subsets U_1 of S_1,…, U_n of S_n, such that for $1\leq i<j\leq n$, we have $P\in U_i$ if and only if $P\in U_j$.

The *initial* star $\boldsymbol{\mathcal{S}}_0$ of (S,P) is defined as follows: for every open subset U of $S(n)$, $\mathcal{O}_{\boldsymbol{\mathcal{S}}_0}(U)$ is the set of $(\alpha_1,\ldots,\alpha_n)\in\mathcal{O}_{S_1}(U\cap S_1)\times\cdots\mathcal{O}_{S_n}(U\cap S_n)$ such that if $P\in U$ then $\alpha_1(P)=\cdots=\alpha_n(P)$.

For every n-star $\boldsymbol{\mathcal{S}}$ of (S,P), there is a unique dominant morphism $\boldsymbol{\mathcal{S}}_0\to\boldsymbol{\mathcal{S}}$ inducing the identity on each component. So $\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ is a subring of $\mathcal{O}_{\boldsymbol{\mathcal{S}}_0,P}$.

Note that (iii) is equivalent to

(iii)' For every $\alpha\in\mathcal{O}_{S,P}$, we have $(\alpha,\ldots,\alpha)\in\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$.

5.1.2. Definition: *An* oblate n-star *(or more simply, an* oblate star*) of (S,P) is an n-star $\boldsymbol{\mathcal{S}}$ such that some neighborhood of P in $\boldsymbol{\mathcal{S}}$ can be embedded in a smooth surface.*

5.1.3. Proposition: *An n-star $\boldsymbol{\mathcal{S}}$ is oblate if and only if* $\pi^{-1}(P)\simeq\mathrm{spec}(\mathbb{C}[X]/(X^n))$.

(cf. prop. 4.1.6).

Let $I\subset\{1,\ldots,n\}$ be a nonempty subset. Let $\boldsymbol{\mathcal{S}}^{(I)}=\bigcup_{i\in I}S_i\subset\boldsymbol{\mathcal{S}}$. If $\boldsymbol{\mathcal{S}}$ is oblate then $\boldsymbol{\mathcal{S}}^{(I)}$ is oblate too.

5.2. Properties of oblate stars

Let $\boldsymbol{\mathcal{S}}$ be an oblate n-star of S. Recall that t denotes a generator of the maximal ideal of P in S. We will denote this generator on $S_i \subset \boldsymbol{\mathcal{S}}$ by t_i. We will also denote by π the element $t \circ \pi$ of the maximal ideal of P in $\boldsymbol{\mathcal{S}}$. Let $\mathcal{I}_P$ be the ideal sheaf of P in $\boldsymbol{\mathcal{S}}$.

We begin with 2-stars:

5.2.1. Proposition: *Suppose that $n = 2$. Then*

1 − *There exists a unique integer $p > 0$ such that $\mathcal{I}_{P,P}/(\pi)$ is generated by the image of $(t_1^p, 0)$.*

2 − *The image of $(0, t_2^p)$ is also a generator of $\mathcal{I}_{P,P}/(\pi)$.*

3 − *$(0, t_2^p)$ (resp. $(t_1^p, 0)$) is a generator of the ideal sheaf of S_1 (resp. S_2) at P.*

4 − *$\mathcal{O}_{S^{(2)},P}$ consists of pairs $(\alpha, \beta) \in \mathcal{O}_{S,P} \times \mathcal{O}_{S,P}$ such that $\alpha - \beta \in (t^p)$.*

(cf. prop. 4.2.1 and 4.2.2).

Now suppose that $n \geq 2$. Let $I = \{i, j\} \subset \{1, \ldots, n\}$, with $i \neq j$. Then $S_i \cup S_j \subset \boldsymbol{\mathcal{S}}$ is a 2-star of S. Hence by proposition 5.2.1 there exists a unique integer $p_{ij} > 0$ such that $\mathcal{I}_{P,P}/(\pi)$ (on $S_i \cup S_j$) is generated by the image of $(t_i^{p_{ij}}, 0)$ (and also by the image of $(0, t_j^{p_{ij}})$). Let $p_{ii} = 0$. Then the symmetric matrix $(p_{ij})_{1 \leq i,j \leq n}$ is called the *spectrum* of $\boldsymbol{\mathcal{S}}$.

There exists an element $v_{ij} = (\nu_m)_{1 \leq m \leq n}$ such that $\nu_i = 0$ and $\nu_j = t_j^{p_{ij}}$. For every integer m such that $1 \leq m \leq n$, $m \neq i, j$, there exists an invertible element $\beta_{ij}^{(m)} \in \mathcal{O}_{S,P}$ such that $\nu_m = \beta_{ij}^{(m)} t_m^{p_{im}}$. Let $\beta_{ij}^{(i)} = 0$, $\beta_{ij}^{(j)} = 1$.

5.2.2. Proposition: *Let $\mathbf{b}_{ij}^{(m)} = \beta_{ij}^{(m)}(P) \in \mathbb{C}$. Then we have, for all integers i, j, k, m, q such that $1 \leq i, j, k, m, q \leq n$, $i \neq j$, $i \neq k$*

$$\mathbf{b}_{ik}^{(m)} \mathbf{b}_{ij}^{(q)} = \mathbf{b}_{ik}^{(q)} \mathbf{b}_{ij}^{(m)}.$$

In particular we have $\mathbf{b}_{ij}^{(m)} = \mathbf{b}_{ik}^{(m)} \mathbf{b}_{ij}^{(k)}$ and $\mathbf{b}_{ij}^{(m)} \mathbf{b}_{im}^{(j)} = 1$.

For all distinct integers i, j, k such that $1 \leq i, j, k \leq n$, we have

$$\mathbf{b}_{ki}^{(j)} = -\mathbf{b}_{ik}^{(j)} \mathbf{b}_{ji}^{(k)} .$$

(cf. prop. 4.3.2 and 4.4.6).

Let p be an integer such that $1 \leq p < n$, and $(i_1, j_1), \ldots, (i_p, j_p)$ p pairs of distinct integers of $\{1, \ldots, n\}$. Then the image of $\prod_{m=1}^{p} \mathbf{v}_{i_m j_m}$ is a generator of $(\mathcal{I}_{P,P}^p + (\pi))/(\mathcal{I}_{P,P}^{p+1} + (\pi))$.

Let $I \subset \{1, \cdots, n\}$ be a nonempty subset, distinct from $\{1, \cdots, n\}$. Let $i \in \{1, \cdots, n\} \backslash I$. Let

$$\mathbf{v}_{I,i} = \prod_{j \in I} \mathbf{v}_{ji} .$$

5.2.3. Proposition: *The ideal sheaf of $\boldsymbol{\mathcal{S}}^{(I)}$ in $\boldsymbol{\mathcal{S}}$ is generated by $\mathbf{v}_{I,i}$ at P.*

(cf. prop. 4.3.3).

Note that if $I=\{1,\cdots,n\}\backslash\{i\}$ then $\mathbf{v}_{I,i|S_j}=0$ if $j\neq i$, and $\mathbf{v}_{I,i|S_i}=t_i^{q_i}$, with $q_i=\displaystyle\sum_{1\leq j\leq n}p_{ij}$.

Let i be an integer such that $1\leq i\leq n$ and $J_i=\{1,\ldots,n\}\backslash\{i\}$. Let $\mathcal{K}_i$ be the image of $\mathcal{O}_{\boldsymbol{\mathcal{S}}}$ in $\prod_{1\leq j\leq n, j\neq i}\mathcal{O}_{S_j}/(t_j^{q_j})$. We can view $\mathcal{K}_i$ as a $\mathbb{C}$-algebra. For every $\alpha=(\alpha_m)\in\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$, let $k_i(\alpha)$ be the image of α in $\mathcal{K}_i$.

5.2.4. Proposition: *There exists a morphism of* $\mathbb{C}$*-algebras*

$$\Psi_i:\mathcal{K}_i\longrightarrow\mathcal{O}_{S_{i,P}}/(t_i^{q_i})$$

such that for every $(\alpha_m)_{1\leq m\leq n, m\neq i}\in\mathcal{O}_{\boldsymbol{\mathcal{S}}^{(J_i)},P}$, $\alpha_i\in\mathcal{O}_{S_{j,P}}$, *we have* $\alpha=(\alpha_m)_{1\leq m\leq n}\in\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ *if and only if* $\Psi_i(k_i(\alpha))=[\alpha_i]_{q_i}$.

(cf. prop. 4.4.1).

The morphism Ψ_i has the following properties:

(i) For every $(\alpha_m)_{1\leq m\leq n, m\neq i}\in\mathcal{O}_{\boldsymbol{\mathcal{S}}^{(J_i)},P}$, we have $\Psi_i(\alpha)(P)=\alpha_m(P)$ for $1\leq m\leq n, m\neq i$.
(ii) We have $\Psi_i((t_m)_{1\leq m\leq n, m\neq i})=t_i$.
(iii) Let $j,k\in\{1,\cdots,n\}$ be such that i,j,k are distinct. Let $\mathbf{w}$ be the image of $\mathbf{v}_{jk}$ in $\mathcal{B}_i$. Then there exists $\lambda\in\mathcal{O}^*_{S_{i,P}}$ such that $\Psi_i(\mathbf{w})=\lambda t_i^{p_{ij}}$.
(iv) Let j be an integer such that $1\leq j\leq n$ and $j\neq i$. Let $\mathbf{w}$ be the image of $\mathbf{v}_{ij}$ in $\mathcal{K}_i x$. Then we have $\ker(\Psi_i)=(\mathbf{w})$.

5.2.5. *Converse* – Let $\boldsymbol{\mathcal{S}}^{[n-1]}$ be a $(n-1)$-star of S, with components $S_1,\ldots,S_{n-1}$, of spectrum $(p_{jk})_{1\leq j,k\leq n-1}$. Let $p_{nj}=p_{jn}$, $1\leq j<n$ be positive integers, and $p_{nn}=0$. For $1\leq j\leq n$, let $q_j=\displaystyle\sum_{1\leq k\leq n}p_{kj}$.

Let S_n be another copy of S. Let $\mathcal{K}_n$ be the image of $\mathcal{O}_{\boldsymbol{\mathcal{S}}^{[n-1]}}$ in $\prod_{1\leq j\leq n-1}\mathcal{O}_{S_j}/(t_j^{q_j})$ and

$$\Psi_n:\mathcal{K}_n\longrightarrow\mathcal{O}_{S_n}/(t_n^{q_n})$$

a morphism of $\mathbb{C}$-algebras satisfying properties (i), (ii), (iii) above. Let $\mathcal{K}$ be the subsheaf of algebras of $\mathcal{O}_{\boldsymbol{\mathcal{S}}_0}$ defined by: $\mathcal{K}=\mathcal{O}_{\boldsymbol{\mathcal{S}}_0}$ on $\boldsymbol{\mathcal{S}}_0\backslash\{P\}$, and for every $\alpha=(\alpha_m)_{1\leq m\leq n}\in\mathcal{O}_{\boldsymbol{\mathcal{S}}_0,P}$, $\alpha\in\mathcal{K}_P$ if and only if $\Psi_n(\alpha')=[\alpha_n]_{q_n}$ (where α' is the image of $(\alpha_m)_{1\leq m\leq n-1}$ in $\mathcal{K}_n$).

It is easy to see that $\mathcal{K}$ is the structural sheaf of an oblate n–star of S.

Let $\mathcal{H}=\prod_{1\leq j\leq n}(t_j^{q_j-1})/(t_j^{q_j})\simeq\mathbb{C}^n$ and $\mathcal{K}$ be the image of $\mathcal{O}_{\boldsymbol{\mathcal{S}}}$ in $\prod_{1\leq j\leq n}\mathcal{O}_{S_j}/(t_j^{q_j})$. We can view $\mathcal{K}$ as a $\mathbb{C}$-algebra. Let $\mathcal{J}=\mathcal{H}\cap\mathcal{K}$.

5.2.6. Proposition: *There exists a unique* $\lambda(\boldsymbol{\mathcal{S}})=(\lambda_1,\ldots,\lambda_n)\in\mathbb{P}_n(\mathbb{C})$ *such that for every* $\mathbf{u}=(u_j)_{1\leq j\leq n}\in\mathcal{H}$, *we have* $\mathbf{u}\in\mathcal{J}$ *if and only if* $\lambda_1u_1+\cdots\lambda_nu_n=0$. *The* λ_i *are all non zero.*

(cf. prop. 4.4.5).

For all distinct integers i, j such that $1 \leq i, j \leq n$, we have

$$\frac{\lambda_i}{\lambda_j} \;=\; -\prod_{1\leq m\leq n, m\neq i,j} \mathbf{b}^{(j)}_{mi}.$$

5.3. Construction of oblate stars of a curve

Consider an oblate $(n-1)$-star of S, $\boldsymbol{\mathcal{S}}^{[n-1]}$, with $n-1$ irreducible components $S_1, \ldots, S_{n-1}$, copies of S. Let $(p^{[n-1]}_{ij})_{1\leq i,j<n}$ be its spectrum. For $1 \leq i < n$, let $q^{[n-1]}_i = \sum_{1\leq j<n} p^{[n-1]}_{ij}$. We denote by $\mathcal{I}^{[n-1]}_P$ the ideal of P in $\mathcal{O}_{\boldsymbol{\mathcal{S}}^{[n-1]},P}$. Let $\lambda(\boldsymbol{\mathcal{S}}^{[n-1]}) = (\lambda_1, \ldots, \lambda_{n-1})$.

Let $p_{1n}, \ldots, p_{n-1,n}$ be positive integers, $q_i = q^{[n-1]}_i + p_{in}$ for $1 \leq i < n$, and $q_n = p_{i1} + \cdots + p_{n-1,n}$. Let $\mathbf{u} \in \mathcal{I}^{[n-1]}_{P,P}$ whose image generates $\mathcal{I}^{[n-1]}_P/((\mathcal{I}^{[n-1]}_P)^2 + (\pi))$, of the form

$$\mathbf{u} \;=\; (\beta_1 t_1^{p_{1n}}, \ldots, \beta_{n-1} t_{n-1}^{p_{n-1,n}}),$$

with $\beta_i \in \mathcal{O}_{S_i,P}$ invertible for $1 \leq i < n$.

Let $\mathcal{K}^{[n-1]}$ be the image of $\mathcal{O}_{\boldsymbol{\mathcal{S}}^{[n-1]}}$ in $\mathcal{O}_{S_1}/(t_1^{q_1}) \times \cdots \times \mathcal{O}_{S_{n-1}}/(t_{n-1}^{q_{n-1}})$. We will also denote by $\mathbf{u}$ the image of $\mathbf{u}$ in $\mathcal{K}^{[n-1]}$. Let $\mathcal{Q} = \mathcal{K}^{[n-1]}/(\mathbf{u})$, $\rho : \mathcal{K}^{[n-1]} \to \mathcal{Q}$ the projection and $t_n = \rho(\pi)$.

5.3.1. Proposition: 1 − *We have* $t_n^{q_n} = 0$.

2 − *We have* $t_n^{q_n-1} = 0$ *if and only if*

$$\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \;=\; 0 \ .$$

We suppose now that $\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0$. *Then*

3 − *For every* $\epsilon \in \mathcal{K}^{[n-1]}$ *such that* $\epsilon(P) \neq 0$, *we have* $t^{q_n-1}\epsilon \notin (\mathbf{u})$.

4 − *For every* $\eta \in \mathcal{K}^{[n-1]}/(\mathbf{u})$, *and every integer* k *such that* $1 \leq k < q_n$, *we have* $t_n^k \eta = 0$ *if and only if* η *is a multiple of* $t_n^{q_n-k}$.

5 − $\mathcal{K}^{[n-1]}/(\mathbf{u})$ *is a flat* $\mathbb{C}[t_n]/(t_n^{q_n})$*-module.*

(cf. prop. 4.5.2).

5.3.2. *Construction of stars of a curve* – Suppose that $\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0$. From proposition 5.3.1, 5-, it is easy to prove, using 5.2.5, that there is a unique oblate n-star $\boldsymbol{\mathcal{S}}$ such that $\boldsymbol{\mathcal{S}}^{[n-1]}$ is the union $\bigcup_{1\leq i\leq n-1} S_i$ in $\boldsymbol{\mathcal{S}}$ and Ψ_n is the quotient map $\mathcal{K}_n = \mathcal{K}^{[n-1]} \to \mathcal{Q}$.

5.4. Morphisms of stars

Recall that if $\boldsymbol{\mathcal{S}}$ is an oblate n-star of S, then we have a canonical inclusion of sheaves of algebras (on the underlying topological space $S(n)$ of $\boldsymbol{\mathcal{S}}$) $\mathcal{O}_{\boldsymbol{\mathcal{S}}} \subset \mathcal{O}_{\boldsymbol{\mathcal{S}}_0}$.

Let $\boldsymbol{\mathcal{S}}$, $\boldsymbol{\mathcal{S}}'$ be oblate n-stars of S, with irreducible components $S_1, \ldots, S_n$, and $f : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{S}}'$ a morphism inducing the identity on all the components. Such a morphism exists if and only if $\mathcal{O}_{\boldsymbol{\mathcal{S}}'} \subset \mathcal{O}_{\boldsymbol{\mathcal{S}}}$, and in this case f is unique and is induced by the previous inclusion. Let (p_{ij}) (resp. (p'_{ij})) be the spectrum of $\boldsymbol{\mathcal{S}}$ (resp. $\boldsymbol{\mathcal{S}}'$).

5.4.1. Proposition: *We have $p_{ij} \leq p'_{ij}$ for $1 \leq i, j \leq n$. If f is not the identity morphism then there exist i, j such that $p_{ij} < p'_{ij}$.*

Proof. Let $I = \{i, j\}$. Then f induces a morphism $\boldsymbol{\mathcal{S}}^{(I)} \to \boldsymbol{\mathcal{S}}'^{(I)}$. So we have $\mathcal{O}_{\boldsymbol{\mathcal{S}}'^{(I)},P} \subset \mathcal{O}_{\boldsymbol{\mathcal{S}}^{(I)},P}$. From proposition 5.2.1, 4-, it follows that $p_{ij} \leq p'_{ij}$.

Suppose now that $p'_{ij} = p_{ij}$ for $1 \leq i, j \leq n$. We must prove that $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}'$, i.e. that $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P} = \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$. This is done by induction on n. For $n = 2$ it is obvious. Suppose that it is true for $n-1$. Let $I = \{1, \ldots, n-1\}$. Then f induces a morphism $f_{n-1} : \boldsymbol{\mathcal{S}}^{(I)} \to \boldsymbol{\mathcal{S}}'^{(I)}$. It follows from the induction hypothesis that $\boldsymbol{\mathcal{S}}^{(I)} = \boldsymbol{\mathcal{S}}'^{(I)}$. Since the integers q_i are the same for $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}}'$, the algebras $\mathcal{K}_n$ for $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}}'$ (cf. proposition 5.2.4) are also the same. Now let $\alpha \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$, and let $\beta \in \mathcal{K}_n$ be the image of α. Let $\alpha' \in \mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$ be such that its image in $\mathcal{K}_n$ is also β. Then $\alpha - \alpha'$ belongs to the ideal generated by the $(0, \ldots, 0, t_i^{q_i}, 0 \ldots, 0)$, $1 \leq i \leq n$, which is included in $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$. Hence $\alpha \in \mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$. □

5.4.2. Lemma: *Suppose that f is not the identity morphism. Then there exist an ideal $\mathcal{I} \subset \mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$ and $u \in \mathcal{I}$, $v \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ such that*

$$u \otimes v \neq 0 \qquad \textit{in} \qquad \mathcal{I} \otimes_{\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}} \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$$

and $uv = 0$.

Proof. Let $q_1 = \sum_{i=1}^{n} p_{1i}$, $q'_1 = \sum_{i=1}^{n} p'_{1i}$. According to proposition 5.4.1 we can assume that $q_1 < q'_1$. Let u be a generator of the ideal of S_1 in $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$ and $\mathcal{I} = (u)$. Let $v = (t_1^{q_1}, 0, \ldots, 0)$. We have $uv = 0$. We have to prove that $u \otimes v \neq 0$. We need only to find an $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$-module M and a $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$-bilinear map

$$\phi : \mathcal{I} \otimes_{\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}} \mathcal{O}_{\boldsymbol{\mathcal{S}},P} \longrightarrow M$$

such that $\phi(u \otimes v) \neq 0$. We take $M = \mathcal{O}_{S_1,P}/(t_1^{q'_1})$, which is a quotient of $\mathcal{O}_{\boldsymbol{\mathcal{S}}'}$. It is easy to verify that

$$\phi : ((\lambda_i)_{1 \leq i \leq n} u, (w_i)_{1 \leq i \leq n}) \longmapsto \lambda_1 w_1 \ (\mathrm{mod}\ t_1^{q'_1})$$

is well defined, bilinear, and that $\phi(u \otimes v) \neq 0$. □

5.4.3. Corollary: *Suppose that f is not the identity morphism. Let Y be an algebraic variety and $g : Y \to S$ a morphism such that $g^* : \mathcal{O}_{\boldsymbol{\mathcal{S}},P} \to \mathcal{O}_{Y,P}$ is injective. Then $f \circ g : Y \to S'$ is not flat.*

Proof. We use the notations of the proof of lemma 5.4.2. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\boldsymbol{\mathcal{S}},P} \arrow[r, "g^*"] \arrow[d, "\lambda_S"] & \mathcal{O}_{Y,P} \arrow[d, "\lambda_Y"] \\
\mathcal{I}\otimes_{\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}}\mathcal{O}_{\boldsymbol{\mathcal{S}},P} \arrow[r, "I_{\mathcal{I}}\otimes g^*"] \arrow[d, "\mu_S"] & \mathcal{I}\otimes_{\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}}\mathcal{O}_{Y,P} \arrow[d, "\mu_Y"] \\
\mathcal{O}_{\boldsymbol{\mathcal{S}},P} \arrow[r, "g^*"] & \mathcal{O}_{Y,P}
\end{tikzcd}$$

where $\lambda_S(\alpha)=u\otimes\alpha$, $\mu_S(u\otimes\alpha)=u\alpha$, and λ_Y,μ_Y are defined similarly. It follows that $\mu_Y(u\otimes g^*v)=0$. We will show that $u\otimes g^*v\neq 0$, and this will imply that $f\circ g$ is not flat. Let $w=(t_1^{q_1'},0,\ldots,0)$. Then we have $\mathcal{I}\simeq\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}/(w)$, and from the exact sequence of $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$-modules $0\to(w)\to\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}\to\mathcal{I}\to 0$ we deduce that $\ker(\lambda_Y)=(w).\mathcal{O}_{Y,P}$. Suppose that $u\otimes g^*v=0$. Then g^*v is a multiple of w: $g^*v=w.a$, for some $a\in\mathcal{O}_{Y,P}$. But we have $w=g^*\pi^{q_1'-q_1}v$. Hence $g^*v.(1-g^*\pi^{q_1'-q_1})=0$. Since $1-g^*\pi^{q_1'-q_1}$ is invertible, we have $g^*v=0$, which is false since g^* is injective. Hence $u\otimes g^*v\neq 0$. $\square$

5.5. Structure of ideals

Let $\boldsymbol{\mathcal{S}}$ be an oblate n-star of S.

5.5.1. Proposition: *Let $\mathcal{I}\subset\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ be a proper ideal. Then*

1 - *There exists a positive integer k such that $k\leq n$ and a filtration by ideals*

$$\{0\}=\mathcal{I}_{k+1}\subset\mathcal{I}_k\subset\cdots\subset\mathcal{I}_1=\mathcal{I}$$

such that, for $1\leq i\leq k$ there exists a positive integer j such that $j\leq n$ and an isomorphism $\mathcal{I}_i/\mathcal{I}_{i+1}\simeq\mathcal{O}_{S_j,P}$ of $\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$-modules.

2 - *If $\mathcal{I}_i/\mathcal{I}_{i+1}\simeq\mathcal{O}_{S_j,P}$, then $\mathcal{I}_{i+1}\subset\mathcal{I}_{S_j}$ and $\mathcal{I}_i\not\subset\mathcal{I}_{S_j}$.*

Proof. We prove **1-** by induction on n. The case $n=1$ is trivial. Suppose that $n>1$ and that the result is true for $n-1$. Let $\mathcal{J}_1$ be the ideal sheaf of $S_1\subset\boldsymbol{\mathcal{S}}$, and $\boldsymbol{\mathcal{S}}'=S_2\cup\cdots\cup S_{n-1}\subset\boldsymbol{\mathcal{S}}$. We can view $\mathcal{J}_1$ as an ideal of $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$. We can suppose that $\mathcal{I}\not\subset\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$, i.e that some element of $\mathcal{I}$ has a nonzero first coordinate. Let m be the smallest positive integer such that $\mathcal{I}$ contains an element u of the form

$$u\ =\ (t^m,\alpha_2,\ldots,\alpha_n)\ .$$

Then every element v of $\mathcal{I}$ can be written as

$$v\ =\ \lambda u+v'\ ,$$

with $\lambda\in\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ and $v'\in\mathcal{J}_1\cap\mathcal{I}$, and the first coordinate of λ is uniquely determined. It follows that $\mathcal{I}/(\mathcal{J}_1\cap\mathcal{I})\simeq\mathcal{O}_{S_1,P}$. We can apply the recurrence hypothesis to the ideal $\mathcal{J}_1\cap\mathcal{I}$ of $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$ and get a filtration of it, from which we deduce the filtration of $\mathcal{I}$. This proves **1-** for n.

Now we prove **2-**. Let $\alpha\in\mathcal{O}_{\boldsymbol{\mathcal{S}},P}\backslash\mathcal{I}_{S_j}$. Let $u\in\mathcal{I}_i$ be over a generator of $\mathcal{I}_i/\mathcal{I}_{i+1}$. Then the image of αu in $\mathcal{I}_i/\mathcal{I}_{i+1}$ is not zero, i.e. $\alpha u\not\in\mathcal{I}_{i+1}$. Hence $\alpha\not\in\mathcal{I}_{i+1}$, and $\mathcal{I}_{i+1}\subset\mathcal{I}_{S_j}$. Let

$v_i = (0, \ldots, 0, t_i^{q_i}, 0, \ldots, 0) \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$. Then the image of $v_i u$ in $\mathcal{I}_i/\mathcal{I}_{i+1}$ is not zero, hence $u \notin \mathcal{I}_{S_j}$ and $\mathcal{I}_i \not\subset \mathcal{I}_{S_j}$. □

5.6. Star associated to a fragmented deformation

We keep the notations of chapter 4. Let $n \geq 2$ be an integer, $\pi : \mathcal{C} \to S$ a fragmented deformation of C_n, and $\mathcal{C}_1, \ldots, \mathcal{C}_n$ the irreducible components of $\mathcal{C}$.

Recall that $S(n)$ is the underlying (Zariski) topological space of any n-star of S. Let $\mathcal{C}^{top}$ be the underlying topological space of $\mathcal{C}$. We have an obvious continuous map $\boldsymbol{\pi} : \mathcal{C}^{top} \to S(n)$. Let $\mathcal{A}_n$ be the sheaf of algebras on $S(n)$ defined by: for every open subset U of $S(n)$, $\mathcal{A}_n(U)$ is the algebra of $(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}}(\boldsymbol{\pi}^{-1}(U))$ such that $\alpha_i \in \mathcal{O}_{S_i}(U \cap S_i)$ for $1 \leq i \leq n$.

According to corollary 4.4.8, for every $x \in C$, $\mathcal{A}_{n,P}$ is the algebra of $(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C},x}$ such that $\alpha_i \in \mathcal{O}_{S,P}$ for $1 \leq i \leq n$.

5.6.1. Proposition: *The sheaf $\mathcal{A}_n$ is the structural sheaf of an oblate n-star of S.*

Proof. By induction on n. The case $n = 1$ is obvious. Suppose that $n > 1$ and that the result is true for $n-1$. Let $\mathcal{C}' = \mathcal{C}_1 \cup \cdots \mathcal{C}_{n-1} \subset \mathcal{C}$, and $\mathcal{A}_{n-1}$ the corresponding oblate $(n-1)$-star of S. Let

$$\Phi_n : \mathcal{B}_n \longrightarrow \mathcal{O}_{\mathcal{C}_n}/(\pi_n^{q_n})$$

be the morphism of proposition 4.4.1. According to proposition 4.6.4, Φ_n induces a morphism

$$\Psi_n : \mathcal{K}_n \longrightarrow \mathcal{O}_{S_n,P}/(t_n^{q_n}) \ .$$

By the definitions of $\mathcal{A}_n$ and Φ_n, if $u = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{S_1,P} \times \cdots \times \mathcal{O}_{S_1,P}$, then $u \in \mathcal{A}_{n,P}$ if and only if $\Psi_n(u') = v$, where u' (resp. v) is the image of u in $\mathcal{K}_n$ (resp. $\mathcal{O}_{S_n,P}/(t_n^{q_n})$). The result follows then from 5.2.5. □

We denote by $\mathcal{S}(\mathcal{C})$ (or more simply $\mathcal{S}$) the oblate n-star corresponding to $\mathcal{A}_n$, so $\mathcal{O}_{\mathcal{S}(\mathcal{C})} = \mathcal{A}_n$. From the definition of $\mathcal{A}_n$ we get a canonical morphism

$$\mathbf{\Pi} : \mathcal{C} \longrightarrow \mathcal{S}$$

such that $\mathbf{\Pi}_{|\mathcal{C}_i} = \pi_i : \mathcal{C}_i \to S_i$ for $1 \leq i \leq n$.

5.6.2. Theorem: *The morphism $\mathbf{\Pi}$ is flat.*

Proof. We need only to prove that $\mathbf{\Pi}$ is flat at any point x of C. Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{S},P}$ be a proper ideal. We have to show that the canonical morphism of $\mathcal{O}_{\mathcal{S},P}$-modules

$$\tau = \tau_{\mathcal{I}} : \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathcal{S},P}} \mathcal{I} \longrightarrow \mathcal{O}_{\mathcal{C},x}$$

is injective. According to proposition 5.5.1 there is a filtration by ideals

$$\{0\} = \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \cdots \subset \mathcal{I}_1 = \mathcal{I}$$

such that, for $1 \leq i \leq k$ there exists a positive integer j such that $j \leq n$ and an isomorphism $\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}$ of $\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$-modules. We will prove the injectivity of τ by induction on k.

Recall that for $1 \leq j \leq n$, $\mathcal{I}_{S_j,P} = \mathcal{I}_{S_j,\mathcal{S},P}$ is a principal ideal, generated by an element u_j which is also a generator of $\mathcal{I}_{\mathcal{C}_j,x} = \mathcal{I}_{\mathcal{C}_j,\mathcal{C},x}$ (cf. corollary 4.3.8 and proposition 5.2.3), and that the only zero coordinate of u_j is the j-th.

Suppose that $k = 1$, so $\mathcal{I}$ is isomorphic to $\mathcal{O}_{S_j,P}$ for some j. Let u be a generator of $\mathcal{I}$ and $w \in \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathcal{S},P}} \mathcal{I}$, that can be written as $w = v \otimes u$, $v \in \mathcal{O}_{\mathcal{C},x}$. Suppose that $\tau(v \otimes u) = vu = 0$. Since $\mathcal{I}$ is annihilated by $\mathcal{I}_{S_j,P}$, we have $\mathcal{I} \subset \big((0,\ldots,0,t_j^{q_j},0,\ldots,0)\big)$. Since $vu = 0$, the j-th component of v is zero, i.e. $v \in \mathcal{I}_{\mathcal{C}_j,x}$. Hence v is a multiple of u_j: $v = \alpha u_j$. We have then

$$\begin{aligned} w &= \alpha u_j \otimes u \\ &= \alpha \otimes u_j u \quad (\text{because } \ u_j \in \mathcal{O}_{\mathcal{S},P}) \\ &= 0 \qquad (\text{because } \ u_j u = 0)\ . \end{aligned}$$

Hence τ is injective.

Suppose that the result is true for $k - 1 \geq 1$ and that the filtration of $\mathcal{I}$ is of length k. According to proposition 5.5.1, **1-**, we have $\mathcal{I}/\mathcal{I}_2 \simeq \mathcal{O}_{S_j,P}$ for some j. Let $u \in \mathcal{I}$ be such that its image in $\mathcal{I}/\mathcal{I}_2$ is a generator, and $w \in \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathcal{S},P}} \mathcal{I}$ such that $\tau(w) = 0$. We can write w as $w = \alpha \otimes v + \beta \otimes u$, with $\alpha, \beta \in \mathcal{O}_{\mathcal{C},x}$ and $v \in \mathcal{I}_2$. Since $\alpha v + \beta u = 0$, we have $\beta u \in \mathcal{O}_{\mathcal{C},x}\mathcal{I}_2$, and $\mathcal{O}_{\mathcal{C},x}\mathcal{I}_2 \subset \mathcal{I}_{\mathcal{C}_j}$ by proposition 5.5.1, **2-**, i.e. the j-th coordinate of βu is zero. By proposition 5.5.1, **2-**, the j-th coordinate of u does not vanish, hence the j-th coordinate of β is zero, i.e. $\beta \in \mathcal{I}_{\mathcal{C}_j}$. Hence β is a multiple of u_j: $\beta = \gamma u_j$. We have then

$$\beta \otimes u = \gamma u_j \otimes u = \gamma \otimes u_j u,$$

and $u_j u \in \mathcal{I}_2$ (because its image in $\mathcal{I}/\mathcal{I}_2$ vanishes). It follows that w is the image of an element w' of $\mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathcal{S},P}} \mathcal{I}_2$. We have $\tau_{\mathcal{I}_2}(w') = 0$, hence by the induction hypothesis $w' = 0$. It follows that we have also $w = 0$. $\square$

5.6.3. Remark: If $\mathcal{S}'$ is an oblate n-star of S, and if $\Pi' : \mathcal{C} \to \mathcal{S}'$ is a flat morphism compatible with the projections to S, then we have $\mathcal{S}' = \mathcal{S}(\mathcal{C})$ and $\Pi' = \Pi$. This is an easy consequence of corollary 5.4.3.

5.6.4. *Converse* - Let $\pi : \mathcal{S} \to S$ be an oblate n-star of S. Let $\mathbf{\Pi} : \mathcal{C} \to \mathcal{S}$ be a flat morphism such that for every closed point $s \in \mathcal{S}$, $\mathbf{\Pi}^{-1}(s)$ is a smooth irreducible projective curve. Let $C = \mathbf{\Pi}^{-1}(P)$ and $\tau = \pi \circ \mathbf{\Pi} : \mathcal{C} \to S$. Then $C_n = \tau^{-1}(P)$ is a primitive multiple curve of multiplicity n and associated smooth curve C, and $\mathcal{C}$ is a fragmented deformation of C_n. This is an easy consequence of proposition 4.1.6.

6. Classification of fragmented deformations of length 2

Let $\pi : \mathcal{C} \to \mathbb{C}$ be a fragmented deformation of length 2. The corresponding double curve C_2 is $\pi^{-1}(0)$. Suppose that the spectrum of $\mathcal{C}$ is $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$. This means that the infinitesimal neighborhoods of order p of C in $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic, i.e. we have an isomorphism of sheaves of algebras on C

$$\Phi : \mathcal{O}_{\mathcal{C}_1}/(\pi_1^p) \longrightarrow \mathcal{O}_{\mathcal{C}_2}/(\pi_2^p) \ ,$$

and for every point x of C, we have

$$\mathcal{O}_{\mathcal{C},x} \ = \ \{(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} \ ; \ \alpha_2 \ (\mathrm{mod}\ \pi_2^p) = \Phi(\alpha_1 \ (\mathrm{mod}\ \pi_1^p))\} \ .$$

Let C_i^k denote the infinitesimal neighborhood of order k of C in $\mathcal{C}_i$, $i = 1, 2$, $k > 0$. It is a primitive multiple curve of multiplicity k and associated smooth curve C, and we have $C_1^p = C_2^p$. Hence C_1^{p+1} and C_2^{p+1} appear as extensions of C_1^p in primitive multiple curves of multiplicity $p+1$. According to [4] and [8] these extensions are classified by $H^1(C, T_C)$ (T_C being the tangent sheaf on C). More precisely, we say that two such extensions D, D' are *isomorphic* if there exists an isomorphism $D \simeq D'$ leaving C_1^p invariant. Then if $\mathcal{H}$ is the set of isomorphism classes of such extensions, a bijection $\lambda : H^1(C, T_C) \to \mathcal{H}$ is defined in [4], such that $\lambda(0) = C_1^{p+1}$.

On the other hand, it follows from [2], [4] that the primitive double curves with associated smooth curve C and associated line bundle $\mathcal{O}_C$ are classified by $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$.

6.0.5. Theorem: *The point of* $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ *corresponding to* C_2 *is* $\mathbb{C}.\lambda^{-1}(C_2^{p+1})$.

Proof. According to [4], there exists an open covering $(U_i)_{i\in I}$ of C such that for $k = 1, 2$, the open subset of C_k^{p+1} corresponding to U_i is isomorphic to $U_i \times \mathrm{spec}(C[t]/(t^{p+1}))$. Here t is π_1 on $\mathcal{C}_1$ and π_2 on $\mathcal{C}_2$. We obtain then cocycles $(\theta_{ij}^{(k)})_{i,j\in I}$, where $\theta_{ij}^{(k)}$ is an automorphism of $U_{ij} \times \mathrm{spec}(C[t]/(t^{p+1}))$. We can also suppose that $\omega_{C|U_i}$ is trivial, for every $i \in I$. Let $dx_{ij} = dx$ be a generator of $\omega_C(U_{ij})$. Since the ideal sheaf of C in C_k^{p+1} is the trivial sheaf on C_k^p, we can write, using the notations of [4], $\theta_{ij}^{(k)} = \phi_{\mu_{ij}^{(k)},1}$, with $\mu_{ij}^{(k)} \in \mathcal{O}_C(U_{ij})[t]/(t^p)$, i.e. for every $\alpha \in \mathcal{O}_C(U_i)$, we have, at the level of regular functions

$$\theta_{ij}^{(k)}(\alpha) \ = \ \sum_{m=0}^{p} \frac{1}{m!}(\mu_{ij}^{(k)} t)^m \frac{d^m\alpha}{dx^m} \ ,$$

and $\theta_{ij}^{(k)}(t) = t$. Since $C_1^p = C_2^p$ we can suppose that $\mu_{ij}^{(1)} \equiv \mu_{ij}^{(2)} \ (\mathrm{mod}\ t^{p-1})$. Hence $\tau_{ij} = \mu_{ij}^{(2)} - \mu_{ij}^{(1)} \in (t^{p-1})/(t^p) \simeq \mathcal{O}_C(U_i)$. The family (τ_{ij}) is (in some sense) a cocycle representing $\lambda^{-1}(C_2^{p+1})$ (cf. [4], [8]).

We have $(\pi_1^{p+1}) + (\pi_2^{p+1}) \subset (\pi)$ in $\mathcal{O}_{\mathcal{C}}$. Hence $C_2 = \pi^{-1}(0)$ is contained in the subscheme Z of $\mathcal{C}$ corresponding to the ideal sheaf $(\pi_1^{p+1}) + (\pi_2^{p+1})$. We have

$$\begin{aligned} \mathcal{O}_Z(U_{ij}) \ &= \ \{(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}_1}(U_{ij})/(t^{p+1}) \times \mathcal{O}_{\mathcal{C}_2}(U_{ij})/(t^{p+1}) \ ; \ \Phi(\alpha_1 \bmod t^p) = \alpha_2 \bmod t^p\} \\ &= \ \{(\alpha_1, \alpha_2) \in \mathcal{O}_C(U_{ij})[t]/(t^{p+1}) \times \mathcal{O}_C(U_{ij})[t]/(t^{p+1}) \ ; \ \alpha_1 \equiv \alpha_2 \bmod t^p\}. \end{aligned}$$

To obtain $\mathcal{O}_{C_2}(U_{ij})$, we have just to quotient by $\pi = (t, t)$, and we obtain

$$\mathcal{O}_{C_2}(U_{ij}) \ = \ \mathcal{O}_Z(U_{ij})/(t, t) \ \simeq \ \mathcal{O}_C(U_{ij})[z]/(z^2) \ ,$$

the last isomorphism being

$$(a_0 + a_1 t + \cdots + a_{p-1}t^{p-1} + \alpha t^p, a_0 + a_1 t + \cdots + a_{p-1}t^{p-1} + \beta t^p) \ \mapsto \ \alpha_0 + (\beta - \alpha)z.$$

Now we can make explicit the automorphism of $\mathcal{O}_C(U_{ij})[z]/(z^2)$ induced by θ_{ij} (these isomorphisms will define the cocycle corresponding to C_2). It is easy to see that this isomorphism is $\phi_{\tau_{ij},1}$, which proves theorem 6.0.5. $\square$

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Institut de Mathématiques de Jussieu, Case 247, 4 place Jussieu, F-75252 Paris, France

E-mail address: `drezet@math.jussieu.fr`